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COMPLETENESS OF THE PAPKOVICH POTENTIALS*

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Introduction. If $\mathbf{u}$ is a displacement field in a linear, homogeneous, isotropic medium occupying a region Δ, then, in the absence of body force, it satisfies

$$\nabla\nabla\cdot\mathbf{u} + (1 - 2\nu)\nabla^2\mathbf{u} = 0, \qquad -1 < \nu < \tfrac{1}{2} \tag{1}$$

everywhere in D, the interior of Δ. ν is Poisson's ratio and Gibbs' vector notation is in force.

Contemporary treatments of the various boundary value problems associated with (1) usually begin with a representation of $\mathbf{u}$ in terms of harmonic functions. In this way, one obtains equivalent functional equations on ∂D, the boundary of D, for the determination of these intermediate variables. Of the available general solutions, the most often used is

$$\mathbf{u} = \nabla(\phi + \mathbf{r}\cdot\boldsymbol{\psi}) - 4(1 - \nu)\boldsymbol{\psi} = \mathsf{P}_m\cdot(\phi, \boldsymbol{\psi}), \qquad m = 4(1 - \nu), \qquad \nabla^2(\phi, \boldsymbol{\psi}) = 0 \tag{2}$$

where $\mathbf{r}$ is the position vector and P_m is the linear differential operator which produces $\mathbf{u}$ from the potentials $(\phi, \boldsymbol{\psi})$. Its appeal is found in the low order of derivatives which appear both in (2) and the resulting form of the traction vector on ∂D. Although (2) is referred to as the Papkovich-Neuber solution, its origins are in the work of V. J. Boussinesq of the last century.

When replacing the displacement by $\mathsf{P}_m\cdot(\phi, \boldsymbol{\psi})$, one must be assured the desired $\mathbf{u}$ is contained among the elements of $\mathsf{P}_m\cdot(\phi, \boldsymbol{\psi})$. Assertions guaranteeing this replacement are called completeness theorems for the representation (2), and such results are now classical. Indeed if ∂D and $\mathbf{u}$ are sufficiently smooth, then there always exist harmonic functions $(\phi, \boldsymbol{\psi})$, regular in D, with a prescribed degree of continuity in Δ, such that (2) is valid (see [1], [2]). Furthermore, if Δ is unbounded and $\mathbf{u}$ decays suitably, then so do the potentials [3].

Inherent in all derivations of (2) is the fact that the range of $(\phi, \boldsymbol{\psi})$ is broader than is necessary to underwrite its completeness. The principal efforts attempting to suppress this redundancy relate to the role of the scalar potential ϕ and whether or not it can be taken zero in (2) without impairing completeness. An excellent review of the history of this problem along with its first correct analysis was given in [4]. Their main result, relevant to the work here, is contained in the following theorem:

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THEOREM 1. *Suppose Δ is star-shaped with respect to the origin which is in D, both ∂D and* $\mathbf{u}$ *are sufficiently smooth, and m is not a positive integer. Then every such* $\mathbf{u}$ *is of the form*

$$\mathbf{u} = \mathsf{P}_m \cdot (0, \boldsymbol{\psi}) \quad \text{in } \Delta \tag{3}$$

where $\boldsymbol{\psi}$ *is harmonic and sufficiently smooth.*

The authors of [4] were motivated to Theorem 1 by the nonuniqueness of a set of potentials corresponding to a given $\mathbf{u}$. With this property they proceed, in a cleverly elementary way, by employing solid harmonic expansions in conjunction with an integration of a partial differential equation along a spherical radius. This ray integration is the source of the star-shaped hypothesis while the restriction on ν is eigenvalue in character.

The main result of this paper is an extension of Theorem 1 to a class of periphractic and/or unbounded regions, and the form of (3) when m is a positive integer. These derive from a formulation similar to the oblique derivative problem of potential theory. Subsequent use of the Fredholm alternatives, as they apply to the resulting singular integral equations, provides the extensions contained in Theorems 4, 5, and 6. Exploiting this nonuniqueness in different ways yields two other completeness theorems which limit the range of the boundary values of the potentials. (This, in effect, is how Theorem 1 could be handled also. However, the fact that $\phi = 0$ on ∂D and its harmonicity in D force it to be zero everywhere.) All of the representations presented are unique in the sense that $\mathsf{P}_m \cdot (\phi, \boldsymbol{\psi}) = 0$ if and only if $(\phi, \boldsymbol{\psi}) = 0$ whenever the potentials meet the requirements of the specific theorems. Uniqueness of (3) is implicit, although no specific mention of it occurs in [4].[1] All of these uniqueness theorems find a limited analogy in the nature of the indeterminacy of the Kolosoff potentials [5] for plane problems.

The conclusion of the paper discusses possible extensions of these ideas to a wider range of associated problems which can be set up in a similar way.

1. Nonuniqueness of $(\phi, \boldsymbol{\psi})$. By setting $\mathbf{u} = 0$ in (2) and operating on the remains, first with the divergence and then with the curl (see [4]), we conclude that the potentials

$$\phi^* = m\Phi - \mathbf{r} \cdot \nabla \Phi, \qquad \boldsymbol{\psi}^* = \nabla \Phi \tag{4}$$

are such that $\mathsf{P}_m \cdot (\phi^*, \boldsymbol{\psi}^*) = 0$, whenever Φ is harmonic. Hence if $\mathbf{u} = \mathsf{P}_m \cdot (\phi', \boldsymbol{\psi}')$, then $\mathbf{u} = \mathsf{P}_m \cdot (\phi, \boldsymbol{\psi})$ where

$$\begin{aligned} \phi &= \phi' + m\Phi - \mathbf{r} \cdot \nabla \Phi && \text{(a)} \\ \boldsymbol{\psi} &= \boldsymbol{\psi}' + \nabla \Phi && \text{(b)} \\ \nabla^2 \Phi &= 0. && \text{(c)} \end{aligned} \tag{5}$$

This redundancy in (2) leads us to three classical boundary value problems of potential theory for a determination of Φ by suitably fixing the boundary values of ϕ and/or $\boldsymbol{\psi}$.

[1]Such information is vital, for example, in finite difference techniques which replace Δ by a rectangular space lattice with p interior nodes and q boundary points. In using (2), there are altogether $4(p + q)$ nodal values of $(\phi, \boldsymbol{\psi})$. The difference approximation of Laplace's equation gives $4p$ relations between these unknowns and the boundary conditions $3q$ more. Any use of a digital computer requires an additional q relations, which is precisely what the uniqueness theorems supply. It should be remarked that individual solutions employing other methods for computation hurdle this facet of (2) on an *ad hoc* basis.

2. Dirichlet's problem for Φ. From (5a, b), we have that

$$\phi + \mathbf{r}\cdot\boldsymbol{\psi} = \phi' + \mathbf{r}\cdot\boldsymbol{\psi}' + m\Phi_1 \tag{6}$$

on ∂D. By specifying $\phi + \mathbf{r}\cdot\boldsymbol{\psi}$ on this surface, the boundary values of Φ_1 are defined by (6) since $(\phi', \boldsymbol{\psi}')$ is known in Δ. Taking this combination to be zero on ∂D, we arrive at

THEOREM 2. *If Δ is a bounded region for which* (2) *holds, then* (2) *is complete when $(\phi, \boldsymbol{\psi})$ is restricted to the subset $(\phi, \boldsymbol{\psi})_1$ for which all elements satisfy $\phi + \mathbf{r}\cdot\boldsymbol{\psi} = 0$ on ∂D. Furthermore $\mathsf{P}_m\cdot(\phi, \boldsymbol{\psi})_1 = 0$ if and only if $(\phi, \boldsymbol{\psi})_1 = 0$ in Δ.*

(6) gives the boundary values of $m\Phi_1$ as $-\phi' - \mathbf{r}\cdot\boldsymbol{\psi}'$, and consequently the harmonic function Φ_1 is uniquely determined in Δ. Thus if $\mathbf{u}$, in (2), results from $(\phi', \boldsymbol{\psi}')$, then according to (5a, b) with $\Phi = \Phi_1$, the set $(\phi, \boldsymbol{\psi})$ also gives $\mathbf{u}$ and is such that $\phi + \mathbf{r}\cdot\boldsymbol{\psi} = 0$ on ∂D. If $(\phi_1, \boldsymbol{\psi}_1)$ are so limited, and $\mathsf{P}_m\cdot(\phi_1, \boldsymbol{\psi}_1) = 0$ in Δ, it follows by taking the divergence of this last relation that $\nabla\cdot\boldsymbol{\psi}_1 = 0$ in Δ. But $\nabla^2(\phi_1 + \mathbf{r}\cdot\boldsymbol{\psi}_1) = 2\nabla\cdot\boldsymbol{\psi}_1$ and therefore $\phi_1 + \mathbf{r}\cdot\boldsymbol{\psi}_1 = 0$ everywhere in Δ. This reduces $\mathsf{P}_m\cdot(\phi_1, \boldsymbol{\psi}_1) = 0$ to $-m\boldsymbol{\psi}_1 = 0$, and the proof of Theorem 2 is complete.

3. Neumann problem for Φ. By dotting (5b) with $\mathbf{n}$, the outer normal to ∂D, we obtain

$$\partial\Phi_2/\partial n = \mathbf{n}\cdot\boldsymbol{\psi} - \mathbf{n}\cdot\boldsymbol{\psi}' \tag{7}$$

as the boundary value of the normal derivative of Φ_2. In order for (7) to give a well-posed Neumann problem, we limit $\boldsymbol{\psi}$ to those functions for which $\mathbf{n}\cdot\boldsymbol{\psi} = C$ on ∂D. The consistency condition $\int_{\partial D} (\partial\Phi_2/\partial n)\, dQ = 0$, fixes the constant C by

$$C = s^{-1}\int_{\partial D} \mathbf{n}\cdot\boldsymbol{\psi}'\, dQ$$

where s is the surface area of ∂D. The integral can be written differently by using (2):

$$\int_{\partial D} \mathbf{n}\cdot\mathbf{u}\, dQ = 2\int_D \nabla\cdot\boldsymbol{\psi}'\, dQ - m\int_{\partial D} \mathbf{n}\cdot\boldsymbol{\psi}'\, dQ = -(m-2)\int_{\partial D} \mathbf{n}\cdot\boldsymbol{\psi}'\, dQ.$$

Hence taking

$$C = -[(m-2)s^{-1}]\int_{\partial D} \mathbf{n}\cdot\mathbf{u}\, dQ, \tag{8}$$

Φ_2 is determined by (7) to within a constant, and we can state

THEOREM 3. *If Δ is a bounded region for which* (2) *holds, then* (2) *is complete when $(\phi, \boldsymbol{\psi})$ is restricted to the subset for which $\mathbf{n}\cdot\boldsymbol{\psi} = C$ on ∂D and $\phi = 0$ at some preassigned point P_0 of Δ. Furthermore the representation is unique.*

Since (7) and (8) fix Φ_2 to within a constant, the relation (5b) evaluated at P_0 removes this indeterminacy. If $\mathsf{P}_m\cdot(\phi_1, \boldsymbol{\psi}_1) = 0$ and $(\phi_1, \boldsymbol{\psi}_1)$ are as in Theorem 3, it follows that $\phi_1 + \mathbf{r}\cdot\boldsymbol{\psi}_1 =$ const. in Δ. Therefore $\boldsymbol{\psi}_1 = 0$ and ϕ_1 is a constant which must be zero. QED.

It is worth calling attention to the fact that this theorem supposes a knowledge of the boundary displacements. When the traction vector, $\mathbf{t}$, is given and the displacement and rotation is assigned at a point, $\mathbf{u}$ is unique. For this situation, the identity

$$\int_{\partial D} \mathbf{r} \cdot \mathbf{t} \, dQ = (3\lambda + 2\mu) \int_D \nabla \cdot \mathbf{u} \, dQ$$

coupled with the fact that $\nabla \cdot \mathbf{u} = (2 - m)\nabla \cdot \boldsymbol{\psi}'$ allows for an evaluation of the constant C.

In closing, we note that in regard to footnote 1, both this and the previous theorem supply the additional q relations for the nodal values of $(\phi, \boldsymbol{\psi})$.

4. Oblique derivative problem for Φ. The remaining possibility is to set $\phi = 0$ in (5a). Unlike the two previous conditions, the vanishing of ϕ on ∂D implies it is zero throughout a bounded Δ. For infinite regions, this is still true provided ϕ decays rapidly enough. Thus this last boundary condition has the potential to resolve the principal problem of redundancy. Especially, we might hope to reveal the character of the convexity restriction of Theorem 1 and whether or not it is intrinsic to the problem. One might conjecture that an investigation employing integral equations could unravel this enigma. To probe this matter, we offer an analysis based on just such an approach. Contrary to the above aspirations, the conventional theorems regarding the applicability of Fredholm's alternatives in the cases to appear here limit ∂D more severely than does Theorem 1. Our arguments are presented with the view that the methodology developed will lead to an eventual resolution of this aspect of Theorem 1.

Setting $\phi = 0$ on ∂D, (5a) becomes

$$\mathbf{r} \cdot \nabla \Phi_3 - m\Phi_3 = \phi' \tag{9}$$

for the boundary condition on Φ_3. This problem can be treated by expressing Φ_3 in the form

$$\Phi_3(P) = \int_{\partial D} \frac{\alpha(Q)}{R(P, Q)} \, dQ, \qquad R(P, Q) = |\mathbf{r}(P) - \mathbf{r}(Q)| \tag{10}$$

and then by determining $\alpha(Q)$ so that (9) is satisfied by (10) on ∂D. Using the known continuity properties of a single layer and its derivatives, we deduce the following singular integral equation for α:

$$2\pi\mathbf{r}(p) \cdot \mathbf{n}(p)\alpha(p) + \int_{\partial D} \alpha(Q)K(p, Q) \, dQ = \phi'(p), \qquad p \in \partial D \tag{11}$$

where $\mathbf{n}$ is the outer normal to ∂D and

$$K(p, Q) = -\mathbf{r}(p) \cdot [\mathbf{r}(p) - \mathbf{r}(Q)] \frac{1}{R^3} - \frac{m}{R}. \tag{12}$$

The Fredholm theorems, [6], apply if and only if

$$\mathbf{r}(p) \cdot \mathbf{n}(p) \quad \text{is never zero for } p \in \partial D, \tag{13}$$

and then (11) is solvable if and only if

$$4\pi \int_{\partial D} \phi'(Q)\beta(Q) \, dQ = 0 \tag{14}$$

where $\beta(Q)$ is the solution of the homogeneous adjoint

$$2\pi\mathbf{r}(p) \cdot \mathbf{n}(p)\beta(p) + \int_{\partial D} \beta(Q)K(Q, p) \, dQ = 0, \qquad p \in \partial D. \tag{15}$$

Calling D^* the complement of Δ and writing

$$\Psi(P) = \int_{\partial D} \frac{\beta(Q)}{R(P, Q)}\, dQ = \begin{matrix} \Psi^-(P) & P \in D^* \\ \Psi^+(P) & P \in D \end{matrix} \tag{16}$$

it follows that at ∂D

$$\partial\Psi^+/\partial n - \partial\Psi^-/\partial n = 4\pi\beta(p) \tag{17}$$

where $\mathbf{n}$ is as in (11). Placing this into (13) we have

$$\int_{\partial D} \phi'\left[\frac{\partial\Psi^+}{\partial n} - \frac{\partial\Psi^-}{\partial n}\right] dQ = \int_{\partial D}\left[\Psi^+\frac{\partial\phi'}{\partial n} - \phi'\frac{\partial\Psi^-}{\partial n}\right] dQ = 0. \tag{18}$$

(Green's second identity, as it applies to ϕ', Ψ^+, has been used to reduce the first integral of (18).)

To compute $\Psi(P)$ we note that if

$$\mathbf{r}\cdot\nabla\Psi + (1 + m)\Psi = 0 \qquad \text{in} \quad D^* + D = \Delta^* \tag{19}$$

then the same arguments which led to the formation of (11) now produce (15) from (16) and (19). Since $\Psi(P)$ is continuous in all space, the solution of (19) gives the ingredients to apply (18). Because of (13), we need only consider (19) for a very special class of D^*. First let S_0, S_1 be two sufficiently smooth star-shaped surfaces with respect to the origin which is inside S_0 which, in turn, is inside S_1 and (13) is satisfied for both. (It is not difficult to generate simple examples meeting these requirements.) With such S_0, S_1 we have

Case (a). D^* is the region D_a^* exterior to S_0. Then if m is the positive integer $n - 1$

$$\Psi^-(P) = H_{-n}(P), \qquad \Psi^+(p) = H_{-n}(p) \tag{20a}$$

where $H_{-n}(P)$ is an arbitrary solid harmonic of negative integral degree. For all other values of m, we have

$$\Psi(P) = 0 \text{ in all of space.} \tag{20b}$$

Case (b). D^* is the region D_b^* interior to S_0. Then

$$\Psi(P) = 0 \quad \text{in all of space} \tag{21}$$

for all values of m.

Case (c). D^* is the region D_0^* interior to S_0 plus D_1^* which is exterior to S_1. In this case

$$\Psi^-(P) = \begin{matrix} 0 & \text{in} \quad D_0^* \\ H_{-n}(P) & \text{in} \quad D_1^* \end{matrix} \tag{22}$$

when m is the positive integer $n - 1$. For all other values of m, which is always positive or zero, $\Psi(P) = 0$ in all of space.

All of these are established in the same way and so we only obtain (20). From (16), $\Psi^-(P)$ is harmonic in D_a^* and vanishes at infinity. Hence for sufficiently large $|\mathbf{r}|$, $\Psi^-(P) = \sum_0^\infty H_{-k}(P)$. The linear independence of solid harmonics implies (20) for these values of $|\mathbf{r}|$. But $\Psi^-(P)$ is regular in all of D_a^*; hence by the principle of analytic continuation for harmonic functions, (20) holds for all $P \epsilon D_a^*$. The argument for nonexceptional values of m goes in a similar way.

With these preliminaries out of the way, we present the main theorems of the paper.

THEOREM 4. *If $D^* = D_a$ and* (2) *holds in Δ, then ϕ may be taken zero in* (2) *without loss of generality provided m is not a positive integer. For the exceptional case* (2) *is complete when ϕ is limited to the set of solid harmonics $H_m(P)$.*

To prove this theorem, we note that for the nonexceptional values (18) and therefore (14) is satisfied for any ϕ'. Consequently (11) always has a solution Φ_3 which meets (9) with $\phi = 0$ on ∂D. Since ϕ is regular and harmonic in D it follows that $\phi = 0$ in Δ. When m is an integer, we write (2) in the form

$$\mathbf{u} = \nabla H_m' + \mathsf{P}_m \cdot (\phi'', \psi) \tag{23}$$

where $\phi'' = \phi' - H_m'$ and H_m' is the solid harmonic of degree m which occurs in an expansion of ϕ' in the vicinity of the origin. If ϕ'' is zero on ∂D, then the solvability condition as given by (18) becomes

$$\int_{\partial D} \left[\frac{\partial \phi''}{\partial n} H_{-(m+1)} - \phi'' \frac{\partial H}{\partial n} - (m + 1) \right] dQ = 0. \tag{24}$$

An expansion of ϕ'' in solid harmonics in a sufficiently small sphere Γ about the origin does not contain any harmonic of degree m. Transferring the surface integral of (24) to the spherical boundary of Γ, it follows that (14) is satisfied because of the orthogonality properties of the surface harmonics. Hence ϕ'' can be taken zero in all of Δ and the proof is complete.

THEOREM 5. *If $D^* = D_b$ and* $\mathbf{u}$ *of* (2) *is such that* (ϕ, ψ) *vanish at infinity, then* (2) *is complete for all m when ϕ is taken zero.*

THEOREM 6. *If $D^* = D_c$ then* $\mathbf{u} = \mathsf{P}_m \cdot (0, \psi)$ *is complete when m is not an integer. If m is an integer and if S_0 , S_1 are such that a spherical surface about the origin can be constructed which is wholly in D, then*

$$\mathbf{u} = \mathsf{P}_m \cdot (H_m , \psi) \tag{25}$$

is complete.

The proofs of the last two theorems only involve the elements of the proof of Theorem 4 and so we dispense with them.

5. Conclusions. The exceptional cases of Theorems 4 and 6 provide the counterexample presented in [4] to show that when $4(1-\nu)$ is a positive integer, (3) could not hold. Actually, the device leading to (25) could also be employed in the proof contained in [4].

It would be desirable to have some test on the boundary data $\mathbf{u}$ to know whether or not the H_m in (25) could be taken zero. If one sets up an argument similar to that of Sec. 4 based on the representation of ψ as a single vector layer, the symbol of the resulting singular vector integral equation vanishes and the Fredholm theorems are not applicable.

Finally it is worth recalling from [4] and also from the nature of the analysis presented here that whatever the shape and connectivity of Δ may be, if D contains the origin and m is a positive integer, then $\mathsf{P}_m \cdot (0, \psi)$ can never be complete when ψ is restricted to the class of regular harmonic functions.

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