POTENTIAL FLOW WHEN A UNIFORM STREAM OF INVISCID LIQUID IS DISTURBED BY AN OVAL OF CASSINI*

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Consider two fixed points A, B where $AB = 2c$. The locus of a point P which moves so that $AP \cdot BP = b^2$, a positive constant, is a curve called an oval of Cassini. When $b \gg c$, the curve is a closed convex oval. As b decreases the curve ceases to be convex, developing a "waist", and finally, when $b = c$, becomes a figure of 8 known as Bernoulli's Lemniscate. If $b < c$ the locus splits into two closed curves, one within each loop of the lemniscate, and finally degenerates into the two points A and B when $b = 0$. We shall be concerned here only with the case $b \geq c$. Taking A and B to be the points $z = -c$ and $z = c$, $z = x + iy$, these curves are given by $\xi = \alpha$, where α is a constant parameter, in the net

$$z = c(1 + e^{2\zeta})^{1/2}, \qquad \zeta = \xi + i\eta, \qquad \alpha \geq 0, \tag{1}$$

the lemniscate corresponding to $\alpha = 0$.

In two-dimensional flow let one of these curves, say $\xi = \alpha$, disturb the uniform stream whose complex potential is $Ue^{-i\beta}z$. The fluid then occupies the region for which $\xi > \alpha$. Therefore of two points for which the real part of ζ is ξ, or $2\alpha - \xi$, only one lies in the region of the flow.

From (1) we have

$$Ue^{-i\beta}z = Ue^{-i\beta}c(1 + e^{2\zeta})^{1/2} = F_1(\zeta) + F_2(\zeta) \tag{2}$$

where, for sufficiently great values of ξ, we have by the binomial theorem

$$F_1(\zeta) = Ue^{-i\beta}ce^{\zeta}, \qquad F_2(\zeta) = Ue^{-i\beta}c(\tfrac{1}{2}e^{-\zeta} - \tfrac{1}{8}e^{-3\zeta} + \cdots). \tag{3}$$

Thus $F_2(\zeta) \to 0$ as $\xi \to \infty$ and the complex potential of the given stream tends to $F_1(\zeta)$. Now since $\bar{f}(\zeta) = \overline{f(\bar{\zeta})}$, where the bar denotes the complex conjugate, we see that $\overline{F}_1(2\alpha - \zeta) = Ue^{i\beta}ce^{2\alpha-\zeta}$ tends to zero when $\xi \to \infty$. Thus by a general method [1] the complex potential for the disturbed flow is

$$w = F_1(\zeta) + \overline{F}_1(2\alpha - \zeta) = 2Uce^{\alpha} \cosh(\zeta - \alpha - i\beta) \tag{4}$$

The verification is immediate, for when $\xi = \alpha$, i.e. on the oval, w is real so that the stream function is zero and the oval is a streamline, while when $\xi \to \infty$, w tends to $F_1(\zeta)$ which is the complex potential of the stream.

We also notice that the only singularity of $F_1(\zeta)$ is at infinity in the flow and therefore $\overline{F}_1(2\alpha - \zeta)$ introduces no new singularities into the flow.

The case of the lemniscate, $\alpha = 0$, is interesting, for this curve has a double point at the origin, where the two tangents intersect at right angles. Thus the streamline

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$\psi = 0$ intersects itself at right angles at the origin which is therefore a stagnation point whatever the value of β, the direction of the stream.

REFERENCE

[1] L. M. Milne-Thomson, *Theoretical hydrodynamics*, Macmillan, London and New York, 1968, Sec. 6.35