AN APPLICATION OF THE METHOD OF MOMENTS TO STOCHASTIC EQUATIONS*

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Abstract. A modified form of Galerkin's method is formally applied to an equation involving a stochastic bounded linear operator. The result, in general, is a sequence of stochastic linear algebraic equations. In the case of a statistically homogeneous operator, however, it is possible to obtain a sequence of deterministic linear algebraic equations. The formalism is applied to determining the electric field in a dielectric with a statistically homogeneous random permittivity.

Introduction. In formulating a wide variety of continuum problems involving a medium with material properties about which one only has limited information, e.g., the permittivity in a dielectric ($\epsilon(\mathbf{r})$), it is frequently convenient to introduce the concept of a random medium. A random medium may be viewed as a family of media together with a probability distribution defined over the members of the family. The more information one possesses of the relevant material properties, the smaller will be the size of the family encompassed by the random medium and/or the more selective will be the probability distribution. Such formulations result in field equations which are random (stochastic) in the same sense as above. The problem is to invert these equations subject to any auxiliary conditions that may be present, e.g., to determine the family of static electric fields existing in the media ($\mathbf{E}(\mathbf{r})$). The stochastic solution so defined is then a family of solutions together with a probability distribution defined over the members.

The carrying out of a solution to the above problem is extremely complicated. In most practical cases it is made impossible by the fact that the probability distribution defined on the family of media involved is not known in detail. Rather, all that is available is some indirect information which is usually expressed in terms of various statistical averages. It is obvious that in such cases, a complete determination of the family of solutions together with a probability distribution to assign to this family is not possible. In the best of situations, it is necessary to be content with obtaining some partial information regarding the solution which likewise is usually in the form of various statistical averages. In most important problems even this is not possible since a desired statistical average of the solution variable is most often not uniquely determined by a finite number of statistical averages of the variable defined by the random medium involved. Complete knowledge of the probability distribution associated with the random medium is required to determine even the simplest statistical average of the solution variable in these cases. Because of this one is led to attempt to look for approximate answers for statistical averages of the solution variable which depend only on a finite number of statistical averages of any parameters which appear in the governing equation.

Two approaches to the problem are readily apparent. The first approach is to manip-

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ulate and average the stochastic field equations so as to obtain deterministic equations on the statistical averages of the solution variable. The result of such an approach is to obtain an infinite set of deterministic equations. An approximation may then be introduced by truncating the set in some way, usually by invoking some physical argument, and the truncated set of deterministic equations is then solved to obtain the desired statistical averages. This approach is most commonly used and several schemes have been introduced for the systematic truncation of the statistical equations. A second approach would be to attempt to obtain a general solution of the stochastic field equation in the sense that the solution to any member of the family of equations defined by the stochastic equation is a special case of the general solution. Once this has been accomplished the statistical averages of the solution variable that are of interest are directly calculable. The difficulty with this second approach is that the stochastic field equations in the problems of interest are usually of the form of differential or integral equations with coefficients which are random functions of the independent variable(s). There are, in general, no methods available for finding the general solution to such problems exactly, and hence one is required to look for approximate techniques if any success is to be achieved. Toward this end perturbation theory has been used to treat those continuum problems involving a random medium which differs only slightly from a given homogeneous medium. The perturbation theory is of only limited validity, however, and in dealing with stochastic equations it is difficult to ascertain just what are the limits of validity.

Another approach which may prove fruitful in obtaining the approximate solution of a differential or integral equation involving stochastic coefficients is to attempt to modify the method first given by Galerkin [1]. Galerkin's method has been of great use in approximately solving differential and integral equations with variable, although deterministic, coefficients. Briefly, Galerkin's method consists first of all in introducing a sequence of ever-increasing subspaces of the Hilbert space on which the problem is defined so that the limit of the sequence will be the entire space. Next a projection operation is introduced and used in conjunction with this sequence of subspaces to define a sequence of problems, the elements of which are the projections of the problem to be solved onto each of the subspaces introduced above. Each of the projected problems so obtained requires for its solution the inversion of a set of linear algebraic equations. Intuitively, one might suspect that the solution of the limit of the sequence of projected problems will converge to the solution of the original problem so long as the original problem is selected from some suitably selected class. Convergence proofs have been carried out, particularly in the Russian literature, for several classes of problems [2], [3], [4], [5], [6]. In the absence of such proofs one is forced to rely on intuition strengthened by the usefulness of results obtained by employing the method.

Assuming that the sequence of projected solutions converges to the desired solution, then each element of the sequence may be viewed as an approximate solution. How good is such an approximation? Clearly the answer to this depends on the sequence of subspaces which is arbitrarily selected in the first step. Herein lies the art in applying Galerkin's method. It is obvious that a good choice for a sequence of subspaces will depend on the problem to be approximately solved. Herein lies the difficulty in attempting to apply Galerkin's method to a problem involving a stochastic equation. If the stochastic nature of the equation is viewed as an ignorance factor, it is this ignorance which makes a useful choice of base elements difficult.

A modification can be introduced into Galerkin's method which eliminates much of the aspect of art in its successful application. This modification is a formal procedure by which the problem to be solved dictates the sequence of subspaces to be used. Such a modification, termed the method of moments, is presented in a book by Vorobyev [7]. While the method of moments is not as widely applicable as is Galerkin's method, i.e., in its simplest form it is valid only for equations involving linear bounded operators, its loss in generality is counterbalanced by its straightforward application for those problems in which it is applicable.

Formal application of the method of moments to a problem involving a stochastic field equation results in a sequence of subspaces, upon which the problem is to be projected, known only in a statistical sense. That is, each member of the family of equations represented by the stochastic equation will result in defining a different sequence of subspace. Similarly, the sequence of projected problems is, in general, also statistical in nature. For a special class of problems, however, it is possible to arrive at a sequence of projected problems which is deterministic. That is, the sequence of problems obtained by projecting a member of the family of equations onto the sequence of subspaces to be associated with this member is independent of the particular member chosen. For this to be true it is necessary that all random functions appearing in the problem be statistically homogeneous, allowing our invoking an ergodic-type hypothesis, i.e., the equating of an ensemble average to an average taken over the independent variable.

In the present paper we begin with a short review of Galerkin's method as applied to two types of problems, indicating the calculations to be performed in carrying out a solution. The method of moments procedure is then introduced and it is shown how its formal application to a class of problems involving stochastic operators results in an iteration scheme involving deterministic sets of linear algebraic equations. The method is then applied to the problem of determining the electric field in a medium with statistically homogeneous random variations in permittivity. Details are carried out for a one-term approximation and the result is compared with previously reported work on this problem.

1. General discussion. Let us begin by considering two problems which may be expressed by the equations

$$Ax = f \tag{1.1}$$

and

$$x = \mu Ax + f. \tag{1.2}$$

In Eqs. (1.1) and (1.2), x is the unknown element and f is the given element of some Hilbert space H, while A is a linear bounded operator defined on H and μ is a parameter. For clarity it might be well to think of A as a differential or integer operator. For the present we take A and f to be uniquely specified so that we are dealing with a deterministic problem. Eq. (1.1) possesses a solution provided zero is not an eigenvalue of the operator A and Eq. (1.2) possesses a solution provided μ^{-1} is not an eigenvalue of A. It might be noted that the solution to the Eq. (1.2) may be given in terms of the classical Liouville-Neumann series

$$x = f + \mu Af + \mu^2 AAf + \cdots + \mu^n \overbrace{A \cdots A}^{n\ \text{times}} f + \cdots \tag{1.3}$$

provided μ is small enough.

In order to apply Galerkin's iterative scheme to either of the above problems we begin by selecting a system $\{\varphi_k\}$ of linearly independent elements φ_k which is suitable to serve as a basis for H. Thus, any element in H is uniquely representable by a convergent series of the form

$$y = \sum_{k=0}^{\infty} b_k \varphi_k \tag{1.4}$$

where the coefficients b_k depend on y. The space defined by the first $n\varphi_k$'s is a subspace of H which we denote by H_n. A projection operator P_n giving the projection of the element y on H_n is defined by simply truncating the series in Eq. (1.4) after n terms, i.e.,

$$P_n y = \sum_{k=0}^{n-1} b_k \varphi_k \,. \tag{1.5}$$

The projection of the problem represented by Eq. (1.2) onto H_n may now be defined by the equation

$$X_n = \mu A_n x_n + f_n \tag{1.6}$$

where x_n and f_n are the projected solution element and forcing element, respectively, and A_n is the projection of the operator A on H_n defined by

$$A_n = P_n A P_n. \tag{1.7}$$

The sequence of solutions $\{x_n;\ n = 0, 1, \cdots\}$ may be shown to converge to the desired solution x provided that A is taken from a suitably restricted class of operators. The sequence of problems defined by equation (1.1) is obvious after the above discussion.

The actual calculations to be performed in order to obtain the solution of Eq. (1.6) are readily obtained. Expanding f_n in terms of the base elements $\{\varphi_k;\ k=0, 1, \cdots, n-1\}$, we have

$$f_n = \sum_{k=0}^{n-1} c_k \varphi_k \tag{1.8}$$

where the coefficients c_k are determined by the set of algebraic equations

$$\sum_{k=0}^{n-1} c_k(\varphi_k\,, \varphi_j) = (f, \varphi_j) \qquad j = 0, \cdots, n-1. \tag{1.9}$$

In Eq. (1.9) the notation (a, b) denotes the scalar product, defined for the space H, of the elements a and b. Similarly the solution x_n is expanded in terms of the base elements, i.e.,

$$x_n = \sum_{k=0}^{n-1} a_k \varphi_k \,. \tag{1.10}$$

Operating on x_n by A_n as defined by Eq. (1.7) and substituting the result into Eq. (1.6), we obtain the following set of algebraic equations on the coefficients A_k :

$$\sum_{k=0}^{n-1} (\delta_{kj} - \mu\, d_{kj}) a_k = C_j \qquad j = 0, \cdots, n-1 \tag{1.11}$$

where d_{kj} is determined by the equations

$$\sum_{j=0}^{n-1} d_{kj}(\varphi_j\,, \varphi_i) = (A\varphi_k\,, \varphi_i) \qquad i = 0, \cdots, n-1. \tag{1.12}$$

The projected problem, therefore, consists in inverting three sets of linear algebraic equations given by Eqs. (1.9), (1.11) and (1.12).

The method of moments iteration scheme differs from the Galerkin scheme in that we begin not by selecting a basis for H but by selecting a single element in H which we may denote by z_0 . Next we generate an infinite sequence of elements $\{z_k\}$ from this arbitrarily chosen element by means of the recurrence relation

$$z_n = Az_{n-1} \quad n = 1, 2, \cdots . \tag{1.13}$$

Assuming that all of the elements so generated are linearly independent, the z_k's, like the φ_k's in Galerkin's method, may be used to define a sequence of ever increasing subspaces. One important difference is that the limit of the subspaces defined by the z_k's, which we denote by H_z , is not necessarily the entire space H. The subspace H_z may be said to reduce the operator A, however, in the sense that the result of operating on any element H_z by A will itself be an element of H_z . Hence, under some circumstances one can show that the solution to the problems given by Eqs. (1.1) or (1.2) will lie in H_z provided we insure that the forcing term lies in H_z . This is easily accomplished by choosing $z_0 = f$. It might be noted that this choice of z_0 results in a sequence of elements $\{z_k\}$ which may be identified, except for a constant, with the terms in the Liouville-Neumann series solution to Eq. (1.2).

Once the z_k's have been selected as indicated, the solution may be carried out in the manner outlined for Galerkin's method. Alternately, one might suspect that the special nature of the z_k's may result in a simple structure for A_n , thereby allowing a somewhat easier determination of the solution of Eq. (1.6). Vorobyev [7] does obtain a solution that would require less work than does the inversion of the three $(n \times n)$ matrices required by Eqs. (1.9), (1.11) and (1.12). The same argument applies to the problem obtained by projecting Eq. (1.1) onto H_n .

We consider now the formal application of the method of moments to the case in which A is not deterministic. The system of base elements obtained by successive application of A will obviously be stochastic in that every member of the family contained in A will define a separate system of base elements and hence a separate sequence of subspaces. The system of projected problems as defined by the sets of algebraic Eqs. given in (1.9), (1.11) and (1.12) will likewise be, in general, stochastic requiring a different solution for each member of the family contained in A. While the solution of a set of random algebraic equations is easier to obtain than is a random differential equation the task would still be formidable. Let us, therefore, not consider the general problem but consider a special class of problems for which the system of projected problems is not stochastic. This will occur even in the case in which the n z_k's and the forcing term f are stochastic so long as the scalar product defined for the space H of any two of these elements is deterministic. In dealing with differential or integral operators which contain random functions of the independent variable(s), i.e., space or time, it is possible, by means of an ergodic-type hypothesis, to define a Hilbert space with a scalar product which we can equate to an ensemble average so long as all random functions are statistically homogeneous. In such cases the scalar product of two random functions will be deterministic.

For an important class of problems involving stochastic operators, therefore, the method of moments provides an iteration technique for obtaining their solution which requires the solution of a sequence of deterministic algebraic problems. In the remainder of the paper the technique is applied to obtain the first term in the sequence of two such problems.

2. Electric field with prescribed average.

2.1 *Formulation of problem.* In this section we should like to treat the problem of determining the static electric field, $\mathbf{E}(\mathbf{x})$, in an infinite medium with permittivity, $\epsilon(\mathbf{x})$, which can only be described in a statistical sense. The media will be taken to be statistically homogeneous and isotropic. Thus, all moments of $\epsilon(\mathbf{x})$ will depend only upon the difference in coordinates and will be independent of their absolute orientation. The solution will be given in terms of a prescribed average value for the electric field which we shall take to be independent of position. Averages are taken here in an ensemble sense but use of an ergodic-type argument allows us to associate such averages with volume averages by virtue of the assumptions of statistical homogeneity which were just prescribed.

The equations governing the electric field are

$$\nabla \cdot (\epsilon \mathbf{E}) = 0, \tag{2.1}$$

and

$$\nabla \times \mathbf{E} = 0. \tag{2.2}$$

Writing

$$\mathbf{E}(\mathbf{x}) = \langle \mathbf{E}_3 \rangle \, \mathbf{k} + \mathbf{E}', \tag{2.3}$$

where $\langle \mathbf{E}_3 \rangle \, \mathbf{k}$ is the prescribed average value of $\mathbf{E}(\mathbf{x})$, and introducing the term

$$\alpha(\mathbf{x}) = \ln \epsilon(\mathbf{x}) - \langle \ln \epsilon(\mathbf{x}) \rangle, \tag{2.4}$$

where the $\langle \quad \rangle$ indicate ensemble averaging, allows us to write the following equations on $\mathbf{E}'$:

$$\nabla \cdot \mathbf{E}' + \langle E_3 \rangle \mathbf{k} \cdot \nabla \alpha + \nabla \alpha \cdot \mathbf{E}' = 0, \tag{2.5}$$

$$\nabla \times \mathbf{E}' = 0 \tag{2.6}$$

Following Prager [8], Eqs. (2.5) and (2.6) may be replaced by an integral equation formulation by means of the free-space Green's function $1/r$. Thus

$$\mathbf{E}'(\mathbf{x}) = -\frac{\langle E_3 \rangle}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \frac{\partial \alpha(\mathbf{x}')}{\partial x_3'} \, dv' - \frac{1}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \nabla' \alpha(\mathbf{x}') \cdot \mathbf{E}'(\mathbf{x}') \, dv' \tag{2.7}$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ and ∇' is the gradient operator taken with respect to $\mathbf{x}'$.

2.2 *Operational formulation.* To express Eq. (2.7) in terms of operational notation we introduce the Hilbert space H consisting of all vector functions $\mathbf{G}(\mathbf{x})$ defined over all values of $\mathbf{x}$ for which the volume integral

$$\lim_{v \to \infty} \frac{1}{v} \int_v \mathbf{G}(\mathbf{x}) \cdot \mathbf{G}(\mathbf{x}) \, dv$$

exists. Addition and multiplication by a scalar are defined according to the usual rules for addition and multiplication of vector functions. The scalar product associated with the pair of elements $\mathbf{G}(\mathbf{x})$ and $\mathbf{H}(\mathbf{r})$ is denoted by $(\mathbf{G}, \mathbf{H})$ and defined by

$$(\mathbf{G}, \mathbf{H}) = \lim_{v\to\infty} \frac{1}{v} \int [\mathbf{G}(\mathbf{x})\cdot\mathbf{H}(\mathbf{x})]\, dv.$$

The norm of an element $\mathbf{G}(\mathbf{x})$, denoted by $||\mathbf{G}||$, is given by

$$||\mathbf{G}|| = \sqrt{\lim_{v\to\infty} \frac{1}{v} \int_v \mathbf{G}(\mathbf{x})\cdot\mathbf{G}(\mathbf{x})\, dv}.$$

If we now consider that the elements in our space are known only in a statistical sense and restrict ourselves to vector functions of position which are statistically homogeneous, then we may associate the scalar product and norm with the following statistical means:

$$(\mathbf{G}, \mathbf{H}) = \langle \mathbf{G}(\mathbf{x})\cdot\mathbf{H}(\mathbf{x})\rangle,$$
$$||\mathbf{G}|| = \sqrt{\langle \mathbf{G}(\mathbf{x})\cdot\mathbf{G}(\mathbf{x})\rangle}.$$

By introducing the operator A defined on our space by

$$A\mathbf{G} = -\frac{1}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \nabla'\alpha(\mathbf{x}')\cdot\mathbf{G}(\mathbf{x}')\, dv', \tag{2.8}$$

the integral equation governing $\mathbf{E}'$ may be written in the form

$$\mathbf{E}' = \mathbf{F} + A\mathbf{E}' \tag{2.9}$$

where

$$\mathbf{F} = -\frac{\langle E_3\rangle}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \frac{\partial\alpha(\mathbf{x}')}{\partial x_3'}\, dv'. \tag{2.10}$$

2.3 *One-term method of moments approximation.* Restricting our attention to those media for which a method of moments iteration converges, we shall now calculate the first-term approximation. The single-term approximation is obtained from a trivial solution of Eqs. (1.9), (1.11) and (1.12) for $n = 1$, once we choose $\varphi_0 = \mathbf{F}(\mathbf{x})$, and may be written simply as

$$\mathbf{E}'(\mathbf{x}) = a_0\, \mathbf{F}(\mathbf{x}), \tag{2.11}$$

where the constant a_0 is given by

$$a_0 = \frac{1}{1 - (A\mathbf{F}, \mathbf{F})/(\mathbf{F}, \mathbf{F})}, \tag{2.12}$$

which for the statistical problem is given by

$$a_0 = \frac{1}{1 - \langle A\mathbf{F}\cdot\mathbf{F}\rangle/\langle\mathbf{F}\cdot\mathbf{F}\rangle}. \tag{2.13}$$

Carrying out the operations indicated by Eq. (2.13), we find it convenient to take the ensemble average at the point $\mathbf{x} = 0$. So doing, we obtain

$$\langle\mathbf{F}\cdot\mathbf{F}\rangle = \frac{\langle E_3\rangle^2}{16\pi^2} \int_{v_r} \int_{v_s} \frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3} \frac{\partial^2}{\partial r_3\, \partial s_3} \langle\alpha(\mathbf{r})\alpha(\mathbf{s})\rangle\, dv_r\, dv_s\, . \tag{2.14}$$

Applying the restriction of statistical homogeneity and isotropy, it is possible to carry out the integrations indicated in Eq. (2.14) giving

$$\langle \mathbf{F}\cdot\mathbf{F}\rangle = \langle E_3\rangle^2 \langle \alpha^2\rangle/3. \tag{2.15}$$

The appendix may be referred to for intermediate calculations.

Operating of $\mathbf{F}$ with A, forming the scalar product of the result with $\mathbf{F}$ and taking the ensemble average at the point $\mathbf{x} = 0$ gives

$$\langle A\mathbf{F}\cdot\mathbf{F}\rangle = -\frac{\langle E_3\rangle^2}{64\pi^3}\int_{v_r}\int_{v_s}\int_{v_t}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{t}}{t^3}\,\frac{(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\cdot\nabla_{\mathbf{r}}\,\frac{\partial^2}{\partial s_3\,\partial t_3}\langle\alpha(\mathbf{r})\alpha(\mathbf{s})\alpha(\mathbf{t})\rangle\,dv_r\,dv_s\,dv_t \tag{2.16}$$

where $\nabla\mathbf{r}$ is the gradient operator with respect to the position vector $\mathbf{r}$ keeping $\mathbf{s}$ and $\mathbf{t}$ constant. Applying the restriction of statistical homogeneity and isotropy to Eq. (2.16) allows a partial integration giving

$$\langle A\mathbf{F}\cdot\mathbf{F}\rangle = -\langle E_3\rangle^2\, I \tag{2.17}$$

where

$$I = \frac{1}{16\pi^2}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\,\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\alpha(0)\alpha(\mathbf{r})\alpha(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{2.18}$$

Substitution of these results into Eq. (2.13) gives

$$a_0 = \frac{1}{1 + 3I/\langle\alpha^2\rangle}. \tag{2.19}$$

2.4 *Comparison with previous results.* Next, we consider a limiting form of the above solution for the case of small fluctuations of the permittivity above some average value. Writing

$$\epsilon(\mathbf{x}) = \langle\epsilon\rangle + \epsilon'(\mathbf{x}), \tag{2.20}$$

this corresponds to $|\epsilon'|/\langle\epsilon\rangle \ll 1$.

Expansion of the various expressions for small $\epsilon'/\langle\epsilon\rangle$ and retention of a single term gives

$$\mathbf{F}(\mathbf{x}) \sim -\frac{\langle E_3\rangle}{4\pi\langle\epsilon\rangle}\int_{v'}\frac{\mathbf{r}}{r^3}\,\frac{\partial\epsilon'(\mathbf{x}')}{\partial x_3'}\,dv' \tag{2.21}$$

and

$$I \sim \frac{1}{16\pi^2\langle\epsilon\rangle^3}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\,\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\epsilon'(0)\epsilon'(\mathbf{r})\epsilon'(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{2.22}$$

Therefore, in the limit of small perturbations, the method of moments result reduces to

$$\mathbf{E}'(\mathbf{x}) = -\frac{\langle E_3\rangle}{4\pi\langle\epsilon\rangle}\int_{v'}\frac{\mathbf{r}}{r^3}\,\frac{\partial\epsilon'(\mathbf{x}')}{\partial x_3'}\,dv'. \tag{2.23}$$

This result agrees with the single-term perturbation solution given by Brown [9], Prager [8], and Beran and Molyneux [10].

In subsequent work dealing with obtaining bounds on an effective permittivity, Beran [11] used a variational principle to obtain an improved solution. The variational principle invoked states that the only vector function $\mathbf{E}(\mathbf{x})$ taken from the class of functions which satisfy $\nabla\times\mathbf{E} = 0$ in some domain together with a prescribed value of $\mathbf{E}$ on the boundary of the domain which will also satisfy the equation $\nabla\cdot\epsilon\mathbf{E} = 0$ in the domain is the one for which the functional

$$U = \frac{1}{2v} \int_v \epsilon \mathbf{E} \cdot \mathbf{E} \, dv,$$

where V is taken over the domain, is a minimum. For the case in which $\epsilon(\mathbf{x})$ is described in a statistical sense, Beran modified the above principle by first replacing the condition that all trial functions $\mathbf{E}(\mathbf{x})$ satisfy a prescribed value on some boundary by a condition that the ensemble average of all trial functions be some prescribed value and also by replacing the volume integral in the functional to be minimized by an ensemble average, i.e.,

$$U = \tfrac{1}{2} \langle \epsilon \mathbf{E} \cdot \mathbf{E} \rangle.$$

This modified principle was then used to ascertain which $\mathbf{E}(\mathbf{x})$ of the family defined by

$$\mathbf{E}(\mathbf{x}) = \langle E_3 \rangle \mathbf{k} + \lambda \mathbf{E}_1(\mathbf{x}),$$

where $\mathbf{E}_1(\mathbf{x})$ is the small perturbation solution given in Eq. (2.23), will minimize the functional given above. The correct value for λ was shown to be [11, Eq. 18]:

$$\lambda = \frac{1}{1 + 3I/\langle \epsilon'^2 \rangle} \tag{2.24}$$

where I is given by the limiting form in Eq. (2.22). It is interesting to note that this value of λ corresponds to that which would be obtained for a_0 if we formally replaced $\langle A\mathbf{F} \cdot \mathbf{F} \rangle / \langle \mathbf{F} \cdot \mathbf{F} \rangle$ by its limit as $|\epsilon'|/\langle \epsilon \rangle \to 0$.

3. Electric displacement vector with prescribed average.

3.1 *Formulation of problem.* A problem closely related to that in Sec. 2 is to obtain the solution in terms of a prescribed average value for the electric displacement vector $\mathbf{D}(\mathbf{y}) = \epsilon(\mathbf{y})\mathbf{E}(\mathbf{y})$. The preceding solution is of little use since it requires prescription of $\langle E \rangle = \langle \mathbf{D}/\epsilon \rangle$ and not $\langle \mathbf{D} \rangle$. The same restrictions to media which are statistically homogeneous and isotropic that were made in Sec. 2 are also to be made here.

The governing equations expressed in terms of the electric displacement, $\mathbf{D}(\mathbf{y})$, and $\mu(\mathbf{y}) = 1/\epsilon(\mathbf{y})$ are

$$\nabla \cdot \mathbf{D} = 0, \tag{3.1}$$

and

$$\nabla \times (\mu \mathbf{D}) = 0. \tag{3.2}$$

Proceeding as before, we write

$$\mathbf{D}(\mathbf{y}) = \langle \mathbf{D}_3 \rangle \mathbf{k} + \mathbf{D}'(\mathbf{y}), \tag{3.3}$$

and introduce the term

$$\beta(\mathbf{y}) = \ln \mu(\mathbf{y}) - \langle \ln \mu(\mathbf{y}) \rangle, \tag{3.4}$$

allowing us to obtain the following equations on $\mathbf{D}'(\mathbf{y})$:

$$\nabla \cdot \mathbf{D}' = 0 \tag{3.5}$$

and

$$\nabla \times \mathbf{D}' + \nabla\beta \times \langle D_3 \rangle \mathbf{k} + \nabla\beta \times \mathbf{D}' = 0. \tag{3.6}$$

Eq. (3.5) is sufficient for us to conclude that the irrotational part of the vector $\mathbf{D}'$ is zero. Thus, $\mathbf{D}' = \nabla \times \mathbf{H}$ where $\nabla \cdot \mathbf{H} = 0$[1]. Upon substitution into Eq. (3.6), we obtain

$$\nabla^2 \mathbf{H} - \nabla\beta \times (D \times \mathbf{H}) = -\langle D_3 \rangle \mathbf{k} \times \nabla\beta. \tag{3.7}$$

Taking the curl of Eq. (3.7) allows the reintroduction of $\mathbf{D}'$ with the resulting equation expressed as

$$\nabla^2(\mathbf{D}' + \langle D_3 \rangle \mathbf{k}\beta) = \langle D_3 \rangle \mathbf{k} \cdot \nabla\nabla\beta + \nabla \times (\nabla\beta \times \mathbf{D}'). \tag{3.8}$$

Eq. (3.8) may be replaced by an integral equation by using the free-space Green's function $1/r$. Thus, we write

$$\mathbf{D}'(\mathbf{y}) = -\langle D_3 \rangle \beta(\mathbf{y})\mathbf{k} + \frac{\langle D_3 \rangle}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \frac{\partial \beta(\mathbf{y}')}{\partial y_3'}\, dv' + \frac{1}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \times [\nabla' \beta(\mathbf{y}') \times \mathbf{D}'(\mathbf{y}')]\, dv' \tag{3.9}$$

where $\mathbf{r} = \mathbf{y} - \mathbf{y}'$ and ∇' is the gradient operator with respect to $\mathbf{y}'$.

The same function space is introduced as was introduced in Sec. 2 allowing our expression of Eq. (3.9) in the following operator form:

$$\mathbf{D}' = \mathbf{P} + B\mathbf{D}', \tag{3.10}$$

where

$$\mathbf{P}(y) = -\langle D_3 \rangle \beta(\mathbf{y})\mathbf{k} + \frac{\langle D_3 \rangle}{4\pi} \int_{v'} \frac{\mathbf{r}}{r^3} \frac{\partial \beta(\mathbf{y}')}{\partial y_3'}\, dv' \tag{3.11}$$

and the operator B is defined by

$$B\mathbf{G} = \frac{1}{4\pi} \int_{v} \frac{\mathbf{r}}{r^3} \times [\nabla' \beta(\mathbf{y}') \times \mathbf{G}(\mathbf{y}')]\, dv', \tag{3.12}$$

where $\mathbf{G}(\mathbf{y})$ is a generic element of our space.

3.2 *One-term method of moments approximation.* As before, we restrict our attention to those media for which the iteration scheme will converge and calculate the first term approximation. The result is

$$\mathbf{D}'(\mathbf{y}) = b_0\, \mathbf{P}(\mathbf{y}) \tag{3.13}$$

where

$$b_0 = \frac{1}{1 - \langle B\mathbf{P} \cdot \mathbf{P} \rangle / \langle \mathbf{P} \cdot \mathbf{P} \rangle}. \tag{3.14}$$

Applying the definition for the scalar product to $\langle \mathbf{P} \cdot \mathbf{P} \rangle$, with the ensemble average taken at the point $\mathbf{y} = 0_1$ results in

$$\langle \mathbf{P} \cdot \mathbf{P} \rangle = \langle D_3 \rangle^2 \langle \beta^2 \rangle + \frac{\langle D_3 \rangle}{2\pi} \mathbf{k} \cdot \int_{v_r} \frac{\mathbf{r}}{r^3} \frac{\partial}{\partial r_3} \langle \beta(0)\beta(\mathbf{r}) \rangle\, dv_r + \frac{\langle D_3 \rangle^2}{16\pi^2} \int_{v_r} \int_{v_s} \frac{\mathbf{r}}{r^3} \cdot \frac{\mathbf{s}}{s^3} \frac{\partial^2}{\partial r_3\, \partial s_3} \langle \beta(\mathbf{r})\beta(\mathbf{s}) \rangle\, dv_r\, dv_s\, . \tag{3.15}$$

[1]Strictly speaking this is not true since Helmholtz's theorem, which is the root of this statement, requires that the sources lie within a finite region. To give a more proper formalism we must take μ' to be zero outside some finite volume and then subsequently allow the volume to become infinite.

If the restriction to statistically homogeneous and isotropic media is invoked, it is possible to carry out each of the integrations in Eq. (3.15). The result is

$$\langle \mathbf{P}\cdot\mathbf{P}\rangle = 2\,\langle D_3\rangle^2\,\langle\beta^2\rangle/3. \tag{3.16}$$

Carrying out the operation $B\mathbf{P}$ gives

$$\begin{aligned} B\mathbf{P} &= \frac{\langle D_3\rangle}{8\pi}\int_{v'}\frac{\mathbf{y}-\mathbf{y}'}{|\mathbf{y}-\mathbf{y}'|^3}\times[\mathbf{k}\times\nabla'\beta^2(\mathbf{y}')]\,dv' \\ &+\frac{\langle D_3\rangle}{16\pi^2}\int_{v'}\int_{v''}\frac{\mathbf{y}-\mathbf{y}'}{|\mathbf{y}-\mathbf{y}'|^3}\times\left\{\nabla'\times\left[\frac{\mathbf{y}'-\mathbf{y}''}{|\mathbf{y}'-\mathbf{y}''|^3}\frac{\partial}{\partial y_3''}\beta(\mathbf{y}')\beta(\mathbf{y}'')\right]\right\}dv'\,dv'' \end{aligned} \tag{3.17}$$

and the scalar product results in

$$\begin{aligned} \langle B\mathbf{P}\cdot\mathbf{P}\rangle &= -\frac{\langle D_3\rangle^2}{8\pi}\mathbf{k}\cdot\int_{v_r}\frac{\mathbf{r}}{r^3}\times[\nabla_r\times\langle\beta^2(\mathbf{r})\beta(0)\rangle\mathbf{k}]\,dv_r \\ &-\frac{\langle D_3\rangle^2}{32\pi^2}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\times\left[\nabla_s\times\frac{\partial}{\partial r_3}\langle\beta(\mathbf{r})\beta^2(\mathbf{s})\rangle\mathbf{k}\right]dv_r\,dv_s \\ &+\frac{\langle D_3\rangle^2}{16\pi^2}\mathbf{k}\cdot\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\times\left\{\nabla_r\times\left[\frac{\mathbf{r}-\mathbf{s}}{|\mathbf{r}-\mathbf{s}|^3}\frac{\partial}{\partial s_3}\langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{s})\rangle\right]\right\}dv_r\,dv_s \\ &+\frac{\langle D_3\rangle^2}{64\pi^3}\int_{v_r}\int_{v_s}\int_{v_t}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\times\left\{\nabla_s\times\left[\frac{(\mathbf{s}-\mathbf{t})}{|\mathbf{s}-\mathbf{t}|^3}\frac{\partial^2}{\partial r_3\,\partial t_3}\langle\beta(\mathbf{r})\beta(\mathbf{s})\beta(\mathbf{t})\rangle\right]\right\}dv_r\,dv_s\,dv_t\,. \end{aligned} \tag{3.18}$$

Applying the assumption of statistical homogeneity and isotropy allows us to greatly reduce Eq. (3.18), the result being

$$\langle B\mathbf{P}\cdot\mathbf{P}\rangle = -\langle D_3\rangle^2\,I, \tag{3.19}$$

where

$$I = \frac{1}{16\pi^2}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{3.20}$$

Substitution of these results into Eq. (3.14) gives

$$b_0 = \frac{1}{1+3I/2\langle\beta^2\rangle}. \tag{3.21}$$

3.4 *Small perturbation solution.* To obtain the small perturbation solution, we write

$$\mu(\mathbf{y}) = \langle\mu\rangle + \mu'(\mathbf{y}), \tag{3.22}$$

expand all terms in powers of $|\mu'|/\langle\mu\rangle$ and retain only the first term. The result is

$$\mathbf{P}(\mathbf{y}) \sim -\frac{\langle D_3\rangle\mu'(\mathbf{y})\mathbf{k}}{\langle\mu\rangle} + \frac{\langle D_3\rangle}{4\pi\langle\mu\rangle}\int_{v_r}\frac{\mathbf{r}}{r^3}\frac{\partial\mu'(\mathbf{y}')}{\partial y_3'}\,dv', \tag{3.23}$$

and

$$I \sim \frac{1}{16\pi^2\langle\mu\rangle^3}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\mu'(0)\mu'(\mathbf{r})\mu'(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{3.24}$$

In the small perturbation limit we can make the equation of

$$\mu'/\langle\mu\rangle = -\langle\epsilon'\rangle/\langle\epsilon\rangle,$$

which upon substitution in the above gives

$$\mathbf{P}(\mathbf{y}) \sim \frac{\langle D_3\rangle\epsilon'(\mathbf{y})\mathbf{k}}{\langle\epsilon\rangle} - \frac{\langle D_3\rangle}{4\pi\langle\epsilon\rangle}\int_{v_r}\frac{\mathbf{r}}{r^3}\frac{\partial\epsilon'(\mathbf{y}')}{\partial y_3'}\,dv', \tag{3.25}$$

and

$$I \sim -\frac{1}{16\pi^2\langle\epsilon\rangle^3}\int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\epsilon'(0)\epsilon'(\mathbf{r})\epsilon'(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{3.26}$$

Thus, in the limit of small perturbations, the method of moments result reduces to

$$\mathbf{D}'(\mathbf{y}) = \frac{\langle D_3\rangle\epsilon'(\mathbf{y})\mathbf{k}}{\langle\epsilon\rangle} - \frac{\langle D_3\rangle}{4\pi\langle\epsilon\rangle}\int_{v_r}\frac{\mathbf{r}}{r^3}\frac{\partial\epsilon'(\mathbf{y}')}{\partial y_3'}\,dv'. \tag{3.27}$$

This last result agrees with the single-term perturbation solution given by Beran [11].

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APPENDIX: REDUCTION OF VARIOUS INTEGRALS

$$I_1 = \int_{v_r}\int_{v_s}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\frac{\partial^2}{\partial r_3\,\partial s_3}\langle\alpha(\mathbf{r})\alpha(\mathbf{s})\rangle\,dv_r\,dv_s\,. \tag{1}$$

Introducing a change in coordinates from (**r**, **s**) to (**r**, **p**=**s**−**r**), we write the two-point correlation function as $\langle\alpha(\mathbf{r})\,\alpha\,(\mathbf{p}+\mathbf{r})\rangle$ which under assumption of statistical homogeneity is a function of **p** alone. Thus

$$I_1 = -\int_{v_p}\frac{\partial^2 f(\mathbf{p})}{\partial p_3^2}\left\{\int_{v_r}\frac{\mathbf{r}+\mathbf{p}}{|\mathbf{r}+\mathbf{p}|^3}\cdot\frac{\mathbf{r}}{r^3}\,dv_r\right\}dv_p\,.$$

The integration over **r** space is carried out first. The region of integration is interpreted as the region without two vanishingly small spheres around $\mathbf{r} = 0$ and $\mathbf{r} = \mathbf{p}$ and within a third sphere with center at the origin, the radius of which shall increase without limit. To carry out the integration we write

$$\begin{aligned} I_2 &= \int_{v_r}\nabla_r\left(\frac{1}{|\mathbf{r}+\mathbf{p}|}\right)\cdot\nabla_r\left(\frac{1}{r}\right)dv_r \\ &= \int_{v_r}\nabla_r\cdot\left[\frac{1}{|\mathbf{r}+\mathbf{p}|}\nabla_r\left(\frac{1}{r}\right)\right]dv_r - \int_{v_r}\frac{1}{|\mathbf{r}+\mathbf{p}|}\nabla^2\left(\frac{1}{r}\right)dv_r\,. \end{aligned}$$

The second integrand vanishes throughout V_r. The first volume integral is converted to a surface integral by means of Green's theorem. The surface over which the integration is to be carried out consists of the three spheres defining V_r:

$$I_2 = -\int_{s_r}\frac{\mathbf{n}\cdot\mathbf{r}}{|\mathbf{r}+\mathbf{p}|\,r^3}\,da_r$$

where **n** represents the outward normal. Only the integral and the surface of the small sphere about $\mathbf{r} = 0$ do not go to zero when the appropriate limits are taken. This integral is readily evaluated giving $I_2 = 4\pi/\mathbf{p}$. Thus,

$$I_1 = -4\pi\int_{v_p}\frac{1}{p}\frac{\partial^2 f(\mathbf{p})}{\partial p_3^2}\,dv_p\,.$$

To evaluate this integral, we can make use of the assumption of statistical isotropy and write

$$I_1 = -\frac{4\pi}{3}\int_{v_p}\frac{1}{p}\,[\nabla^2 f(p)]\,dv_p\ .$$

Two successive applications of Green's theorem as above allows an immediate evaluation of the above integral giving

$$I_1 = \frac{16\pi^2}{3}\,f(0) = \frac{16\pi^2}{3}\,\langle\alpha^2\rangle,$$
$$I_1 = \int_{v_r}\int_{v_s}\int_{v_t}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{t}}{t^3}\,\frac{(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\cdot\nabla_{\mathbf{r}}\,\frac{\partial^2}{\partial s_3\,\partial t_3}\,\langle\alpha(\mathbf{r})\alpha(\mathbf{s})\alpha(\mathbf{t})\rangle\,dv_r\,dv_s\,dv_t\ . \tag{2}$$

Introducing a change in coordinates from $(\mathbf{r}, \mathbf{s}, \mathbf{t})$ to $(\mathbf{r}, \mathbf{p} = \mathbf{r} - \mathbf{s}, \mathbf{q} = \mathbf{r} - \mathbf{t})$, we write the three-point correlation function as $\langle\alpha(\mathbf{r})\ \alpha\ (\mathbf{r} - \mathbf{p})\ \alpha\ (\mathbf{r} - \mathbf{q})\rangle$ which under assumption of statistical homogeneity is a function of $\mathbf{p}$ and $\mathbf{q}$. Therefore

$$I_1 = \int_{v_p}\int_{v_q}\frac{\mathbf{p}}{p^3}\cdot(\nabla_{\mathbf{p}} + \nabla_{\mathbf{q}})\,\frac{\partial^2 f(\mathbf{p},\mathbf{q})}{\partial p_3\,\partial q_3}\left\{\int_{v_r}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{r}-\mathbf{q}}{|\mathbf{r}-\mathbf{q}|^3}\,dv_r\right\}dv_p\,dv_q\ .$$

Integrating over $\mathbf{r}$ space and rearranging, we may write

$$I_1 = 4\pi\int_{v_q}\frac{1}{q}\,\frac{\partial}{\partial q_3}\left[\int_{v_p}\frac{\mathbf{p}}{p^3}\cdot\nabla_{\mathbf{p}}\,\frac{\partial f(\mathbf{p},\mathbf{q})}{\partial p_3}\,dv_p\right]dv_q$$
$$+\ 4\pi\int_{v_p}\frac{\mathbf{p}}{p^3}\,\frac{\partial}{\partial p_3}\cdot\left[\int_{v_q}\frac{1}{q}\,\nabla_{\mathbf{q}}\,\frac{\partial f(\mathbf{p},\mathbf{q})}{\partial q_3}\,dv_q\right]dv_p\ .$$

Use of Green's theorem allows the integration over $\mathbf{p}$ space for the first integral. The result is seen to vanish since by assumption of statistical isotropy $\partial f(\mathbf{p}, \mathbf{q})/\partial p_3$ must vanish at $\mathbf{p} = 0$. Green's theorem may also be used to simplify the second integral with the result

$$I_1 = 4\pi\int_{v_p}\int_{v_q}\frac{\mathbf{p}}{p^3}\cdot\frac{\mathbf{q}}{q^3}\,\frac{\partial^2 f(\mathbf{p},\mathbf{q})}{\partial p_3\,\partial q_3}\,dv_p\,dv_q\ ,$$
$$I_1 = \int_{v_r}\frac{\mathbf{r}}{r^3}\cdot\mathbf{k}\,\frac{\partial}{\partial r_3}\,\langle\beta(0)\beta(\mathbf{r})\rangle\,dv_r\ . \tag{3}$$

Under the assumption of statistical isotropy, we may write

$$I_1 = \frac{1}{3}\int_{v_r}\frac{\mathbf{r}}{r^3}\cdot\nabla_{\mathbf{r}}\langle\beta(0)\beta(\mathbf{r})\rangle\,dv_r$$

which is readily integrated using Green's theorem. The result is

$$I_1 = -\frac{4\pi\langle\beta^2\rangle}{3}$$

$$I_1 = \mathbf{k}\cdot\int_{v_r}\frac{\mathbf{r}}{r^3}\times[\nabla_{\mathbf{r}}\times\langle\beta^2(\mathbf{r})\beta(0)\rangle\mathbf{k}]\,dv_r$$
$$= -\mathbf{k}\cdot\int_{v_r}\nabla_{\mathbf{r}}\left(\frac{1}{r}\right)\times[\nabla_{\mathbf{r}}\times\langle\beta^2(\mathbf{r})\beta(0)\rangle\mathbf{k}]\,dv_r$$

$$= -\mathbf{k}\cdot\int_{v_r} \nabla_r \times \left[\frac{1}{r}\nabla_r \times \langle\beta^2(\mathbf{r})\beta(0)\rangle\mathbf{k}\right] dv_r$$

$$+ \mathbf{k}\cdot\int_{v_r} \frac{1}{r}\nabla_r \times [\nabla_r \times \langle\beta^2(\mathbf{r})\beta(0)\rangle\mathbf{k}]\, dv_r\,. \tag{4}$$

The first integral may be converted into a surface integral which may be shown to vanish. To evaluate the second we may first expand the integrand to obtain

$$I_1 = \mathbf{k}\cdot\int_{v_r} \frac{1}{r}[\mathbf{k}\cdot\nabla_r\nabla_r - \mathbf{k}\nabla_r^2]\langle\beta^2(\mathbf{r})\beta(0)\rangle\, dv_r$$

$$= \int_{v_r} \frac{1}{r}\left[\frac{\partial^2}{\partial r_3^2} - \nabla_r^2\right]\langle\beta^2(\mathbf{r})\beta(0)\rangle\, dv_r\,.$$

Under the assumption of statistical isotropy this becomes

$$I_1 = -\frac{2}{3}\int_{v_r} \frac{1}{r}\nabla^2\langle\beta^2(\mathbf{r})\beta(0)\rangle\, dv_r$$

which is readily evaluated:

$$I_1 = \frac{8\pi}{3}\langle\beta^3\rangle$$

$$I_1 = \int_{v_r}\int_{v_s} \frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3} \times \left[\nabla_s \times \frac{\partial}{\partial r_3}\langle\beta(\mathbf{r})\beta^2(\mathbf{s})\rangle\mathbf{k}\right] dv_r\, dv_s\,. \tag{5}$$

Introducing a change in coordinates from $(\mathbf{r}, \mathbf{s})$ to $(\mathbf{r}, \mathbf{p}=\mathbf{s}-\mathbf{r})$, we write the two-point correlation function as $\langle\beta(\mathbf{r})\beta^2(\mathbf{p}+\mathbf{r})\rangle$ which under the assumption of statistical homogeneity is a function of $\mathbf{p}$ alone. Thus

$$I_1 = -\int_{v_r}\int_{v_p} \frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{p}+\mathbf{r}}{|\mathbf{p}+\mathbf{r}|^3} \times \left[\nabla_p \times \frac{\partial f(\mathbf{p})\mathbf{p}}{\partial p_3}\right] dv_p\, dv_r$$

$$= \int_{v_p} \left[\nabla_p \times \mathbf{k}\frac{\partial f(\mathbf{p})}{\partial p_3}\right]\cdot\left\{\int_{v_r} \frac{\mathbf{r}}{r^3} \times \frac{(\mathbf{r}+\mathbf{p})}{|\mathbf{r}+\mathbf{p}|^3}\, dv_r\right\} dv_p\,.$$

The integration over $\mathbf{r}$ space can be carried out in a similar manner as was the integral containing the dot product. In the present case the result is zero. Thus

$$I_1 = 0$$

$$I_1 = \mathbf{k}\cdot\int_{v_r}\int_{v_s} \frac{\mathbf{r}}{r^3} \times \left\{\nabla_r \times \left[\frac{(\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3}\frac{\partial}{\partial s_3}\langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{s})\rangle\right]\right\} dv_r\, dv_s\,. \tag{6}$$

Introduce a coordinate transformation: $\mathbf{r} = \mathbf{r}$, $\mathbf{p} = \mathbf{r} - \mathbf{s}$. Thus, we write

$$I_1 = -\mathbf{k}\cdot\int_{v_r}\int_{v_p} \frac{\mathbf{r}}{r^3} \times \left\{(\nabla_r + \nabla_p) \times \left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p}, \mathbf{r})}{\partial p_3}\right]\right\} dv_r\, dv_p$$

where

$$f(\mathbf{p}, \mathbf{r}) = \langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{r}-\mathbf{p})\rangle.$$

Consider first

$$I_2 = -\mathbf{k}\cdot\int_{v_r} \frac{\mathbf{r}}{r^3} \times \left\{\int_{v_p} \nabla_p \times \left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p}, \mathbf{r})}{\partial p_3}\right] dv_p\right\} dv_r\,.$$

The integration over **p** space vanishes. Therefore,

$$I_1 = -\mathbf{k}\cdot\int_{v_r}\int_{v_p}\frac{\mathbf{r}}{r^3}\times\left\{\nabla_r\times\left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial p_3}\right]\right\}dv_p\,dv_r\ .$$

Next consider

$$\begin{aligned}I_3 &= -\mathbf{k}\cdot\int_{v_r}\frac{\mathbf{r}}{r^3}\times\left\{\nabla_r\times\left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial p_s}\right]\right\}dv_r\\ &= \mathbf{k}\cdot\nabla_p\times\left\{\frac{1}{r}\nabla_r\times\left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial p_3}\right]dv_r - \mathbf{k}\cdot\int_{v_r}\frac{1}{r}\nabla_r\times\left\{\nabla_r\times\left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial p_s}\right]\right\}dv_r\ .\end{aligned}$$

The first integral may be converted into a surface integral which can be shown to vanish. Therefore

$$I_3 = -\int_{v_r}\frac{1}{r}\nabla_p\cdot\left[\frac{\mathbf{p}}{p^3}\frac{\partial^2 f(\mathbf{p},\mathbf{r})}{\partial p_3\,\partial r_3}\right]dv_r + \mathbf{k}\cdot\int_{v_r}\frac{1}{r}\nabla_r^2\left[\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},\mathbf{r})}{\partial p_3}\right]dv_r\ .$$

The first integral may be integrated by parts and the second may be evaluated. The result is

$$I_3 = -\int_{v_r}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{p}}{p^3}\frac{\partial^2 f(\mathbf{p},\mathbf{r})}{\partial r_3\,\partial p_3}dv_r - 4\pi\mathbf{k}\cdot\frac{\mathbf{p}}{p^3}\frac{\partial f(\mathbf{p},0)}{\partial p_3}.$$

Therefore

$$I_1 = -\int_{v_r}\int_{v_p}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{p}}{p^3}\frac{\partial^2}{\partial r_3\,\partial p_s}\langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{r}-\mathbf{p})\rangle\,dv_r\,dv_p - 4\pi\int_{v_p}\frac{\mathbf{p}}{p^3}\cdot\mathbf{k}\frac{\partial}{\partial p_3}\langle\beta^2(0)\beta(-\mathbf{p})\rangle\,dv_p.$$

The second term may be evaluated under the assumption of statistical isotropy which allows us to write it as

$$I_4 = -\frac{4\pi}{3}\int_{v_p}\frac{\mathbf{p}}{p^3}\cdot\nabla_p\langle\beta^2(0)\beta(\mathbf{p})\rangle\,dv_p = \frac{16\pi^2}{3}\langle\beta^3\rangle.$$

Therefore

$$\begin{aligned}I_1 &= \frac{16\pi^2}{3}\langle\beta^3\rangle - \int_{v_r}\int_{v_p}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{p}}{p^3}\frac{\partial^2}{\partial r_3\,\partial p_3}\langle\beta(0)\beta(\mathbf{r})\beta(\mathbf{p})\rangle\,dv_r\,dv_p\ ,\\ I_1 &= \int_{v_r}\int_{v_s}\int_{v_t}\frac{\mathbf{r}}{r^3}\cdot\frac{\mathbf{s}}{s^3}\times\left\{\nabla_s\times\left[\frac{\mathbf{s}-\mathbf{t}}{|\mathbf{s}-\mathbf{t}|^3}\frac{\partial^2}{\partial r_3\,\partial t_3}\langle\beta(\mathbf{r})\beta(\mathbf{s})\beta(\mathbf{t})\rangle\right]\right\}dv_r\,dv_s\,dv_t\ .\end{aligned}\tag{7}$$

Introducing a change in coordinates from (**r**, **s**, **t**) to (**p**=**s**−**r**, **s**, **q**=**s**−**t**), we write the three-point correlation function as $\langle\varphi(\mathbf{s}-\mathbf{p})\ \varphi(\mathbf{s})\ \varphi(\mathbf{s}-\mathbf{q})\rangle$ which under the assumption of statistical homogeneity is a function of p and q. Therefore

$$\begin{aligned}I_1 &= \int_{v_r}\int_{v_s}\int_{v_t}\frac{\mathbf{s}-\mathbf{p}}{|\mathbf{s}-\mathbf{p}|^3}\cdot\frac{\mathbf{s}}{s^3}\times\left\{(\nabla_p+\nabla_q)\times\left[\frac{\mathbf{q}}{q^3}\frac{\partial^2 f(\mathbf{p},\mathbf{q})}{\partial p_3\,\partial q_3}\right]\right\}dv_s\,dv_p\,dv_q\\ &= \int_{v_p}\int_{v_q}(\nabla_p+\nabla_q)\times\left[\frac{\mathbf{q}}{q^3}\frac{\partial^2 f(\mathbf{p},\mathbf{q})}{\partial p_3\,\partial q_3}\right]\cdot\left\{\int_{v_s}\frac{\mathbf{s}-\mathbf{p}}{|\mathbf{s}-\mathbf{p}|^3}\times\frac{\mathbf{s}}{s^3}\,dv_s\right\}dv_p\,dv_q\ .\end{aligned}$$

The integration over **s** space can now be carried out and the result is zero.

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