LIMITS ON POSSIBLE SOLUTIONS OF VAN DER POL'S EQUATION*

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Shen (1) has presented a method of constructing a solution of Van der Pol's differential equation in the form of a Weierstrassian elliptic function. Unfortunately, he omits to prove that the function he constructs actually satisfies the equation: in fact it cannot do so and this attractive idea must be given up.

To see the impossibility of any such solution it is simplest to proceed direct from the original equation (Shen's equation (1))

$$(d^2x/dt^2) + \mu(x^2 - 1)(dx/dt) + k^2x = 0 \tag{1}$$

to the corresponding equation in the phase plane:

$$y(dy/dx) + \mu(x^2 - 1)y + k^2x = 0 \tag{2}$$

where $y = dx/dt$. In order that the elliptic function $x = \wp(t)$ should satisfy (1), it is necessary that the algebraic expression

$$y = (4x^3 - g_2x - g_3)^{1/2} \tag{3}$$

should satisfy (2). Shen's argument appears to suppose that it is sufficient for this to be so that $-\mu^2 g_3 = \frac{1}{2}g_2^2$, but this is incorrect, as can be verified by direct substitution. From (3) we have $y(dy/dx) = 6x^2 - \frac{1}{2}g_2$, and so from (2)

$$\mu^2(x^2 - 1)^2(4x^3 - g_2x - g_3) = (6x^2 + k^2x - \tfrac{1}{2}g_2)^2. \tag{4}$$

For the solution to hold along any arc, however small, it is necessary that (4) should be an identity, which is evidently impossible since the left side is of degree 7 and the right of degree 4.

We can go farther and show that no algebraic relation whatsoever between x and y will provide a solution to (2). Such a relation is given by an equation

$$F(x, y) \equiv \sum_{r=0}^{n} y^r f_r(x) = 0 \tag{5}$$

where $f_r(x)$ is a polynomial in x. This can satisfy (2) only if

$$y(\partial F/\partial x) \equiv [\mu(x^2 - 1)y + k^2x]\, \partial F/\partial y. \tag{6}$$

Substituting the power series from (5) into (6) we find that $f_n(x)$ must be a constant and that the degree of $f_{n-r}(x)$ must be $3r$. On the other hand, it is also necessary that $f_1(x)$ should be identically zero. This contradiction shows the impossibility that any such solution should exist.

References

[1] Z. Shen, *A solution of Van der Pol's differential equation*, Quart. Appl. Math. **25**, 299–301 (1967)

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