ON THE ROTATION AND TRANSLATION OF AN INCOMPRESSIBLE MEDIUM IN PLANE MOTION*

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Consider the plane motion of an incompressible fluid in an inertial frame, denoted by I. If this motion be dynamically possible, then

$$\operatorname{div} \mathbf{S}_I + \rho \mathbf{f} = \rho \mathbf{a}_I , \tag{1}$$

$$\operatorname{div} \mathbf{V} = 0 \tag{2}$$

are satisfied. In (1), $\mathbf{S}_I$ is the stress tensor, ρ the density, $\mathbf{f}$ is the body force per unit mass, and $\mathbf{a}_I$ is the acceleration. In (2), $\mathbf{V}$ is the velocity field.

Consider the plane motion in a second frame, denoted by II, which is obtained from I by a three-dimensional translation and a superposed rigid rotation ω = const. normal to the plane of motion. If the translation is dependent on time only, but independent of the spatial co-ordinates, then by the principle of frame indifference [1, Sec. 19], the extra stresses in the fluid are equal in I and II; or

$$\mathbf{S}_I + p_I \mathbf{I} = \mathbf{S}_{II} + p_{II} \mathbf{I}, \tag{3}$$

where p_I and p_{II} are the pressures to be determined from the equations of motion. The equation of motion for II is

$$\operatorname{div} \mathbf{S}_{II} + \rho \mathbf{f} = \rho \mathbf{a}_{II} , \tag{4}$$

where

$$\mathbf{a}_{II} = \mathbf{a}_I - \mathbf{g} - \mathbf{b}''. \tag{5}$$

In (5), $\mathbf{g}$ is the acceleration due to the centrifugal and Coriolis effects and $\mathbf{b}''$ is the acceleration due to translation. Now, (4) is satisfied because in this particular motion, $\mathbf{g}$ is derivable from a potential [2] and also because $\mathbf{b}''$ is. For $\mathbf{b}''$ is independent of co-ordinates and hence

$$\operatorname{Curl}_{II} \mathbf{b}'' = 0, \tag{6}$$

and thus $\mathbf{b}''$ is the gradient of a scalar. The result we have obtained is stronger than the earlier conclusions of Oldroyd and Thomas [3] and Noll [4], for these authors omitted the translation term.

Let $b'' = \operatorname{grad} \phi$, where ϕ is a scalar. Then it is easily seen that only the radial and axial components of $\mathbf{b}''$ can exist or $\phi = rb_r(t) + zb_z(t)$, where r and z are measured in the system II, with $b_r(t)$ and $b_z(t)$ representing the radial and axial components of $\mathbf{b}''$ therein. Thus [4]

$$p_{II} = p_I + \rho(2\omega\psi + \tfrac{1}{2}\omega^2 r^2 + rb_r + zb_z) \tag{7}$$

where ψ is the stream function.[1]

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[1]See [4] for restrictions on the regions for the existence of a single valued potential ψ.

Since the result obtained here is applicable to solids as well [3], [4], we can state it as the following: if a plane motion of an incompressible medium be dynamically possible in an inertial frame of reference, the same motion is also possible in any other frame of reference which is rotating with a constant angular velocity about an axis normal to the plane of motion and also when this frame is subject to a time dependent, but spatially independent, translation such that only the radial and axial components of the translational acceleration exists in the moving frame. This result is equivalent to the introduction of a conservative body force [1, Sec. 30], but does not seem to have been given this interpretation before.

The present result has an interesting consequence for the experiment of Taylor [5], [6]. For we have shown that this can also be performed on a platform moving vertically up or down as the resulting forces on the cylinder vanish in such a vertical motion.

References

[1] C. Truesdell and W. Noll, Handbuch der Physik, Bd III/3, Springer-Verlag, Berlin, 1965
[2] C. Truesdell and R.A. Toupin, Handbuch der Physik, Bd III/1, Springer-Verlag, Berlin, 1960,§207
[3] J. G. Oldroyd and R. H. Thomas, Quart. J. Mech. Appl. Math. **9**, 136-139 (1956)
[4] W. Noll, Quart. Appl. Math. **12**, 317-319 (1957)
[5] G. I. Taylor, Proc. Roy. Soc. Lond. **A93**, 99-113 (1916)
[6] G. I. Taylor, Proc. Roy. Soc. Lond. **A100**, 114-121 (1921)