EXACT SOLUTION OF THE EQUATIONS FOR SHALLOW SHELLS OF REVOLUTION*

By THOMAS J. LARDNER, *Massachusetts Institute of Technology*

Introduction. The purpose of this note is to obtain the exact solution of the equations governing the linear deformation of a particular class of shallow shells of revolution. Of interest in order to gain insight into the behavior of shallow shells is the nature of the exact solution for large values of the argument which we obtain by representing the solution in terms of the G function introduced by Meijer [7]. We discuss the properties of the solution and compare the asymptotic expansion of the exact solution with an approximate asymptotic integration procedure of the differential equations discussed by Reissner [9].

The method of solution discussed here is also applicable to the fourth order equations governing the linear axisymmetric deformation of circular cylindrical shells of variable thickness and anticlastic bending of thin strips [10].

Differential equations. The governing equations for the linear deformation of shallow shells of revolution in the absence of surface loads are known, see, e.g., [1, 3], and may be written in the following form

$$D\nabla^2\nabla^2 w = LF, \qquad A\nabla^2\nabla^2 F = -Lw \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{2}$$

$$L = \frac{d^2 z}{dr^2}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) + \frac{1}{r}\frac{dz}{dr}\frac{\partial^2}{\partial r^2} \tag{3}$$

and $z = z(r)$ is the equation of the middle surface. We consider the class of shallow shells

$$dz/dr = \alpha(r/a)^s \tag{4}$$

where α, s are constants and a is the radius at which $z = 0$.

If we let

$$\rho = r/a, \quad \mu^2 = \alpha a/(AD)^{1/2}, \quad w = a^2 w_0 \tilde{g}, \quad F = a^2 f_0 \tilde{f}, \quad w_0 = (A/D)^{1/2} f_0 \tag{5}$$

Eq. (1), upon use of (4) and (5), can be written in the form

$$\rho^4 \nabla^2\nabla^2 \tilde{\phi} - i\mu^2 \rho^{s+1} N\tilde{\phi} = 0 \tag{6}$$

where

$$N\tilde{\phi} = \rho^2 \tilde{\phi}'' + s\rho\tilde{\phi}' + s\ddot{\tilde{\phi}}, \qquad \tilde{\phi} = \tilde{f} + i\tilde{g}$$

and primes and dots indicate differentiation with respect to ρ and θ, respectively. We

*Received April 5, 1967; revised manuscript received June 29, 1967.

consider solutions of (6) in the form

$$\tilde{\phi}(\rho, \theta) = \phi(\rho) \cos n\theta \tag{7}$$

so that Eq. (6) can be written as follows:

$$\rho^4\nabla^2\nabla^2\phi - i\mu^2\rho^{s+1}(\rho^2\phi'' + s\rho\phi' - sn^2\phi) = 0. \tag{8}$$

For values of $s = 1, 0, -1$ in (4) corresponding to a spherical, conical and logarithmic shell, the solutions of (8) are known [2]–[6].

It was noted by McIlroy [4] that Eq. (8) can be written as a generalized hypergeometric equation[1] of the form

$$\left[\left(\delta - \frac{n}{s+1}\right)\left(\delta + \frac{n}{s+1}\right)\left(\delta - \frac{n+2}{s+1}\right)\left(\delta + \frac{n-2}{s+1}\right) - x\left(\delta^2 + \frac{s-1}{s+1}\,\delta - \frac{sn^2}{(s+1)^2}\right)\right]\phi = 0 \tag{9}$$

where

$$x = i\mu^2(s+1)^{-2}\rho^{s+1}, \qquad \delta = x(d/dx).$$

For the cone $s = 0$, McIlroy obtained the four linearly independent solutions of (9) about the regular singular point $x = 0$ by the method of Frobenius and found that two of the solutions had finite series representations. The remaining two infinite series solutions were found by use of the Euler transform solution of (9) to be expressible in terms of derivatives of Bessel functions.

For our discussion of the solution, Eq. (9) will be written in the form

$$\left[\prod_{j=1}^{4} (\delta - b_j) - x \prod_{j=1}^{2} (\delta - a_j + 1)\right]\phi = 0 \tag{10}$$

where

$$b_1 = -b_2 = n/(s+1), \qquad b_3 = (n+2)/(s+1), \qquad b_4 = -(n-2)/(s+1)$$

$$a_{1,2} = 1 - \frac{1}{2}\frac{s-1}{s+1} \pm \frac{1}{s+1}\left(\left(\frac{s-1}{2}\right)^2 + sn^2\right)^{1/2}$$

Solutions. We first derive the fundamental set of solutions of (10) about the regular singular point $x = 0$ in terms of the generalized hypergeometric function by the method of Frobenius. An integral representation is then introduced which is equivalent to the hypergeometric function solutions. This integral representation is expressed in terms of the G function introduced by Meijer [7] (Erdelyi [8]). We then consider four more integral representations which are a specialization of the general results obtained by Meijer and show that these integrals are solutions to the differential equation (10). These four solutions are considered as the four linearly independent solutions about the point $x = \infty$. The connection formulae [7] expressing the relations between the solutions about $x = \infty$ and the solutions about $x = 0$ are not considered. Instead, we investigate the asymptotic expansions of the solutions about $x = \infty$ and discuss the results in view of the expected physical behavior of shallow shells.

[1]McIlroy observes ([4, p. 9]) that this form was first obtained by M. W. Johnson (unpublished).

Applying the method of Frobenius, we find the four linearly independent solutions of (10) about the regular singular point in terms of the generalized hypergeometric function in the form, $i = 1, \cdots, 4$

$$\phi = x^{b_i} {}_2F_3 \left[\begin{matrix} 1 + b_i - a_1 , \quad 1 + b_i - a_2 \\ 1 + b_i - b_1 , \cdots , * , \cdots , 1 + b_i - b_4 ; x \end{matrix} \right] \tag{11}$$

where * indicates the term $1 + b_i - b_i$ is omitted. We assume in obtaining the solutions (11) that $b_i - b_j \neq 0, \pm 1, \cdots$ $(i \neq j)$, $i, j = 1, \cdots, 4$; otherwise logarithmic terms appear in the solution. Because the main objective of this note is the asymptotic form of the solutions to the differential equation (10) we will not consider solutions involving logarithms. The requirement of no logarithmic solutions is seen upon writing out the solutions (11) to be equivalent to the requirement that $2/(s + 1) \neq N$, $N = \pm 1, \pm 2, \cdots$. We also exclude for a given value of $2/(s + 1) \neq N$ those values of n for which the solution (11) fails. We note that among the cases excluded are those for $s = 0, 1, -1$ corresponding to the cone, the sphere, and the logshell, all of which have been treated previously.

We now introduce the G function due to Meijer [7] which can be related to the generalized hypergeometric function. In this note we consider only those G functions applicable as solutions to (10);[2] the complete discussion of the general case is found in the papers of Meijer.

Consider the four integrals, $i = 1, \cdots, 4$

$$G^{1,2}_{2,4}\left[-x \,\middle|\, \begin{matrix} a_1 , a_2 \\ b_i , b_1 , \cdots , * , \cdots , b_4 \end{matrix} \right] = \frac{1}{2\pi i} \int_C \frac{\Gamma(b_i - s) \prod_{j=1}^{2} \Gamma(1 - a_j + s)}{\prod_{j=1}^{4}{}' \, \Gamma(1 - b_j + s)} (-x)^s \, ds \tag{12}$$

where the contour C runs from $\infty - i\tau$ to $\infty + i\tau$ and encloses all the poles of $\Gamma(b_i - s)$ but none of the poles of $\Gamma(1 - a_j + s)$, and where we assume $a_j - b_i \neq 1, 2 \cdots$, for $j = 1, 2$; $i = 1, \cdots, 4$. A prime on the product symbol $\prod$ indicates that the term $\Gamma(1 - b_i + s)$ is omitted and * indicates the term b_i is omitted in the sequence $b_1, \cdots, b_4$.

Upon evaluation of (12) using Cauchy's residue theorem, we find

$$G^{1,2}_{2,4}\left[-x \,\middle|\, \begin{matrix} a_1 , a_2 \\ b_i , b_1 , \cdots , * , \cdots , b_4 \end{matrix} \right]$$

$$= \frac{\prod_{j=1}^{2} \Gamma(1 - a_j + b_i)}{\prod_{j=1}^{4}{}' \, \Gamma(1 - b_j + b_i)} (-x)^{b_i} {}_2F_3 \left[\begin{matrix} 1 + b_i - a_1 , \quad 1 + b_i - a_2 \\ 1 + b_i - b_1 , \cdots , * , \cdots , 1 + b_i - b_4 ; x \end{matrix} \right]. \tag{13}$$

Therefore, if

$$b_i - b_j \neq 0, \pm 1, \cdots ; \qquad i, j = 1, \cdots, 4; \qquad (i \neq j)$$

the four linearly independent solutions of (10) are given by the four functions $G^{1,2}_{2,4}$ defined by (12).

[2]In all that follows $\arg x = \pi/2$.

In order to treat asymptotic solutions of (10) in a simple manner we now consider four additional solutions of (10) in the form

$$G_{2,4}^{4,1}\left[-x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3, b_4 \end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_1 + s)}{\Gamma(a_2 - s)} (-x)^s \, ds, \tag{14}$$

$$G_{2,4}^{4,1}\left[-x \,\middle|\, \begin{matrix} a_2, a_1 \\ b_1, b_2, b_3, b_4 \end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_2 + s)}{\Gamma(a_1 - s)} (-x)^s \, ds, \tag{15}$$

$$G_{2,4}^{4,0}\left[x \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3, b_4 \end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{4} \Gamma(b_j - s)}{\prod_{j=1}^{2} \Gamma(a_j - s)} x^s \, ds, \tag{16}$$

$$G_{2,4}^{4,0}\left[x \exp(-2\pi i) \,\middle|\, \begin{matrix} a_1, a_2 \\ b_1, b_2, b_3, b_4 \end{matrix}\right] = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{4} \Gamma(b_j - s)}{\prod_{j=1}^{2} \Gamma(a_j - s)} (x \exp(-2\pi i))^s \, ds. \tag{17}$$

where as before the contour C runs from $\infty - i\tau$ to $\infty + i\tau$ and encloses all the poles of $\Gamma(b_j - s)$.

It can be shown easily that the integrals given by (14) to (17) are solutions of the differential equation (10). For example, upon substitution of (14) into the left-hand side of (10) we obtain

$$\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{4} (s - b_j) \prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_1 + s)}{\Gamma(a_2 - s)} (-x)^s \, ds$$
$$+ \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{2} (s - a_j + 1) \prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_1 + s)}{\Gamma(a_2 - s)} (-x)^{s-1} \, ds. \tag{18}$$

We rewrite the first integral of (18) in the form

$$\frac{1}{2\pi i}\int_{C'} \frac{\prod_{j=1}^{4} (s + 1 - b_j) \prod_{j=1}^{4} \Gamma(b_j - s - 1)\Gamma(1 - a_1 + s + 1)}{\Gamma(a_2 - s - 1)} (-x)^{s-1} \, ds \tag{19}$$

where C' is C moved a unit distance to the right parallel to real axis.

Upon rewriting (19) we find

$$\frac{1}{2\pi i}\int_{C'} \frac{(1 - a_1 + s) \prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_1 + s)}{(a_2 - s - 1)^{-1}\Gamma(a_2 - s)} (-x)^{s+1} \, ds. \tag{20}$$

Hence upon substitution into (18), we find

$$\int_C - \int_{C'} \left\{ \frac{\prod_{j=1}^{2} (s - a_j + 1) \prod_{j=1}^{4} \Gamma(b_j - s)\Gamma(1 - a_1 + s)}{\Gamma(a_2 - s)} (-x)^{s+1} \right\} ds. \tag{21}$$

But the difference of the integrals about C and C' is an integral about a closed path in which the integrand is analytic and thus the integral is zero. Therefore Eq. (14) is a solution of the differential equation (10). In a similar way the functions defined by (15) to (17) can be shown to be solutions of (10).

Since the differential equation (10) is a fourth order equation, there are only four linearly independent solutions. Therefore it is possible to express the functions defined in (14) to (17) as linear combinations of the four solutions (12) about $x = 0$. These relations can be found in Meijer's papers [7] and will not be discussed. The advantage of the use of the G function representation of the solutions of the differential equation over the generalized hypergeometric function representation is that the behavior of the solutions (14) to (17) for large values of $|x|$ are expressible in a simple form.

For $|x|$ large, the asymptotic forms of the G function have been discussed by Meijer and are immediately applicable. We find for $|x| \to \infty$, that the solution functions given by (14) and (15) have asymptotic expansions which behave algebraically ([7, p. 232])

$$\begin{aligned} G_{2,4}^{4,1}\left(-x \,\middle|\, \begin{matrix} a_1\ ,\ a_2 \\ \cdots \end{matrix}\right) &\sim (-x)^{a_1-1}\{1 + O(1/|x|) + \cdots\} \\ G_{2,4}^{4,1}\left(-x \,\middle|\, \begin{matrix} a_2\ ,\ a_1 \\ \cdots \end{matrix}\right) &\sim (-x)^{a_2-1}\{1 + O(1/|x|) + \cdots\} \end{aligned} \tag{22}$$

while the solution functions (16) and (17) for $|x| \to \infty$ behave exponentially ([7, p. 234)]

$$\begin{aligned} G_{2,4}^{4,0}(x \mid \cdots) &\sim x^{\gamma} \exp(-2x^{1/2})\{1 + O(1/|x|^{1/2}) + \cdots\} \\ G_{2,4}^{4,0}(xe^{-2\pi i} \mid \cdots) &\sim x^{\gamma} \exp(2x^{1/2})\{1 + O(1/|x|^{1/2}) + \cdots\} \end{aligned} \tag{23}$$

where

$$\gamma = \tfrac{1}{4}((1 - 3s)/(1 + s))$$

and where the higher order terms may be found in [7].

It is of interest to note that the leading terms in the asymptotic expansions (22, 23) for the cone $s = 0$, an excluded case in our discussion, agree with the asymptotic forms of the solution obtained by McIlroy [4].

Concluding remarks. We have obtained four linearly independent solutions of the differential equation (10) about $x = \infty$ which are given by (14) to (17) with asymptotic expansions (22) and (23), as well as the four linearly independent solutions about $x = 0$ which are given by (11).

It can easily be shown that the above solutions are consistent with the expected physical nature of the solutions to the shallow shell equations. First with $x = i\mu^2(s + 1)^{-2}\rho^{s+1}$, if $s > 0$ and $\rho \to 0$, the solutions about $x = 0$, (11), are of the nature of the flat plate solutions. If $-1 < s < 0$ and $\rho \to \infty$, the solutions (16, 17) and (23) give edge zone type solutions while the solutions (14, 15) and (22) give interior type solutions. The term interior solution means the membrane-inextensional bending solution which is the solution of (8) with $\mu = \infty$ corresponding to $D = 0$ (membrane solution) or $A = 0$ (inextensional-bending solution). If $s < -1$ and $\rho \to \infty$ the solutions about $x = 0$ again approach the flat plate solutions.

Alternatively, we can consider the solutions (14) to (17) with asymptotic expansions (22, 23) as the solutions for $\mu \gg 1$, $\rho = O(1)$, which are the solutions applicable to shell problems where $\mu = O(a/h)$. For $\mu \gg 1$, $\rho = O(1)$, the shallow shell equations for arbitrary shells of revolution were solved using an approximate asymptotic integration procedure by Reissner [9]. If the results of [9] are specialized to the class of shallow shells (4) treated here, we find that the asymptotic expansions of the exact solutions given by (22) and (23) agree with interior and edge zone solution contributions found by Reissner [9].

Acknowledgment. The work presented in this paper was supported by the Office of Naval Research under a contract with the Massachusetts Institute of Technology.

References

[1] E. Reissner, *On some problems in shell theory,* Proc. First Symp. on Naval Structural Mech. (1958), Pergamon Press, New York, N. Y., 1960, pp. 74–114

[2] E. Reissner, *On the determination of stresses and displacements for unsymmetric deformations of shallow spherical shells,* J. Math and Phys. **38**, 16–35 (1959)

[3] V. Z. Vlasov, *General theory of shells and its applications in engineering.* English translation NASA TT f-99 April 1964 of Russian edition 1949

[4] M. D. McIlroy, *On the solutions of the differential equations of conical shells,* Ph. D. Dissertation, Massachusetts Institute of Technology, 1958

[5] M. D. McIlroy, *Linear deformations of conical shells,* J. Aero/Space Sciences **26**, 253–254 (1959)

[6] T. J. Lardner, *On the stress distribution in a shallow logarithmic shell of revolution,* J. Math and Phys. **45**, 23–34 (1966)

[7] C. C. Meijer, *On the G-function,* Nederl. Akad. Wetensh., Proc. **49**, 227–237, 344–456, 457–469, 632–641, 765–772, 936–943, 1063–1072, 1165–1175 (1946)

[8] A. Erdelyi et al., *Higher transcendental functions* **1**, McGraw-Hill, New York, 1953

[9] E. Reissner., *On asymptotic solutions for nonsymmetric deformations of shallow shells of revolution,* Int. J. Engnr. Sci. **2**, 27–43 (1964)

[10] H. D. Conway and K. A. Farnham, *Anticlastic curvature of strips of variable thickness,* Int. J. Mech. Sci. **7**, 451–458 (1965)