# — NOTES —

## TWO ELECTROMAGNETIC ANALOGIES FOR A HYDRODYNAMIC PROBLEM*

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**Summary.** It is shown that the equations governing long gravity waves on a rotating earth are the same as the electromagnetic equations in an isotropic plasma on the one hand and ferrites on the other under the influence of a magnetic field.

**1. Long gravity waves on a rotating earth.** The equations associated with the classical long wave theory are well known and have been given by Proudman [3]. They are of the form

$$(\nabla^2 + k^2)\zeta = 0, \tag{1}$$

$$\partial\zeta/\partial n + ip(\partial\zeta/\partial s) = 0, \qquad 0 < p < 1. \tag{2}$$

$\zeta$ is the elevation; $p = \Omega/\omega$, there being a time variation $\exp\{i\omega t\}$; $\Omega$ is the Coriolis parameter. $2\omega_0 \sin \alpha$, $\omega_0$ being the angular velocity of the earth and $\alpha$ the north latitude; $k^2 = (\omega^2 - \Omega^2)gh$, $h$ being the sea depth assumed uniform; $\nabla^2$ is here the two-dimensional Laplacian.

Equation (2) represents the condition that there is zero fluid velocity across a boundary.

**2. An anisotropic plasma problem.** Under a time variation $\exp\{iwt\}$, the electromagnetic field in a homogeneous, weakly ionized, electrically neutral plasma, under the influence of a static magnetic field in the $z$ direction, is given by [1]

$$\nabla \times \mathrm{E} = -i\omega\mu_0\mathrm{H}, \tag{3}$$

$$\nabla \times \mathrm{H} = i\omega[\epsilon]\epsilon_0\mathrm{E}. \tag{4}$$

E, H are the electric and magnetic fields respectively, $\epsilon_0$ , $\mu_0$ the electromagnetic constants of free space and $\epsilon$ is the permittivity matrix

$$[\epsilon] = \begin{vmatrix} \epsilon_1, & -i\epsilon_2, & 0 \\ i\epsilon_2, & \epsilon_1 & 0 \\ 0, & 0 & \epsilon_z \end{vmatrix}, \tag{5}$$

$\epsilon_1$ , $\epsilon_2$ , $\epsilon_3$ are dimensionless quantities.

It is not necessary to discuss the actual values of $\epsilon_1$ , $\epsilon_2$ , $\epsilon_3$ . Suffice it to say that $\epsilon_2$ is proportional to the applied magnetostatic field and changes sign with a change in its direction. Jull [1] showed that solutions exist to Eqs. (3) and (4), independent of $z$, such that $E_z$ , $H_x$ , $H_y$ are all zero and that

$$E_x = \frac{\epsilon_1}{i\omega\epsilon_0\epsilon^*}\frac{\partial H_z}{\partial y} - \frac{\epsilon_2}{\omega\epsilon_0\epsilon^*}\frac{\partial H_z}{\partial x}, \tag{6a}$$

*Received February 3, 1967.

$$E_y = \frac{-\epsilon_1}{i\omega\epsilon_0\epsilon^*}\frac{\partial H_z}{\partial x} - \frac{\epsilon_2}{\omega\epsilon_0\epsilon^*}\frac{\partial H_z}{\partial y} \tag{6b}$$

$$\epsilon^* = \epsilon_1^2 - \epsilon_2^2 . \tag{6c}$$

$H_z$ obeys the two-dimensional wave equation

$$(\nabla^2 + k^2)H_z = 0, \tag{7a}$$

$$k^2 = \omega^2\mu_0\epsilon_0\frac{\epsilon^*}{\epsilon_1}. \tag{7b}$$

Suppose now that there is a perfectly conducting cylindrical boundary, parallel to the $z$ direction. The boundary condition on this is that $E_s$ must vanish, where $E_s$ is the tangential component of the electric field. From Eqs. (6a) and (6b), it follows without any difficulty that

$$E_s = -\frac{\epsilon_1}{i\omega\epsilon_0\epsilon^*}\left(\frac{\partial H_z}{\partial n} - \frac{i\epsilon_2}{\epsilon_1}\frac{\partial H_z}{\partial s}\right)$$

and hence that the boundary condition is defined by

$$\partial H_z/\partial n + ip(\partial H_z/\partial s) = 0, \qquad p = \epsilon_2/\epsilon_1 . \tag{8}$$

**3. The ferrite problem.** Under a time variation exp $\{iwt\}$, the electromagnetic field in a ferrite under the influence of a static magnetic field in the $z$ direction is given by [2]. (Slight changes in notation have been made.)

$$\nabla \times \mathrm{E} = -i\omega[\mu]\mu_0\mathrm{H}, \tag{9a}$$

$$\nabla \times \mathrm{H} = i\omega\epsilon\cdot\mathrm{E}. \tag{9b}$$

$\mu$ is the relative permeability matrix, $\mu_0$ is the permeability of free space and $\epsilon$ is the dielectric constant of the medium.

$$[\mu] = \begin{bmatrix} \mu_1 , & -i\mu_2 & 0 \\ i\mu_2 , & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \tag{10}$$

$\mu_1$ , $\mu_2$ , $\mu_3$ are dimensionless quantities whose actual values are unimportant. Suffice it to say that $\mu_2$ is proportional to the applied magnetostatic field and changes sign with a change in its direction. Kales [2] obtained solutions of Eqs. (9), and it can easily be seen from these that a particular solution exists, independent of $z$, with $H_z$ , $E_x$ , $E_y$ all zero and

$$H_x = -\frac{\mu_1}{i\omega\mu_0\mu^*}\frac{\partial E_z}{\partial y} + \frac{\mu_2}{\omega\mu_0\mu^*}\frac{\partial E_z}{\partial x} , \tag{11a}$$

$$H_y = \frac{\mu_1}{i\omega\mu_0\mu^*}\frac{\partial E_z}{\partial x} + \frac{\mu_2}{\omega\mu_0\mu^*}\frac{\partial E_z}{\partial y} , \tag{11b}$$

$$\mu^* = \mu_1^2 - \mu_2^2 . \tag{11c}$$

$E_z$ obeys the two-dimensional wave equation

$$(\nabla^2 + k^2)E_z = 0, \tag{12a}$$

$$k^2 = \omega^2\epsilon\mu_0\mu^*/\mu_1 . \tag{12b}$$

Suppose now that there is a perfectly conducting cylindrical boundary, parallel to the $z$ direction. A boundary condition on this is that $H_n$ must vanish, where $H_n$ is the normal component of the magnetic field. From Eqs. (11a) and (11b) it follows without any difficulty that

$$H_n = -\frac{\mu_1}{i\omega\mu_0\mu^*}\frac{\partial E_z}{\partial s} + \frac{\mu_2}{\omega\mu_0\mu^*}\frac{\partial E_z}{\partial n}$$

and hence that the boundary condition is

$$\partial E_z/\partial n + ip\,\partial E_z/\partial s = 0, \qquad p = \mu_1/\mu_2\,. \tag{13}$$

**4. Discussion.** It will be seen that the problems defined by the sets of Eqs. (1) and (2), (7a) and (8), (12a) and (13) are mathematically formally identical. This suggests that solutions to the longwave problems with complicated boundaries could be solved by experimental techniques using either the plasma or ferrite analogies. In particular, because $\mu_2$, $\epsilon_2$ are proportional to the magnetostatic field (which can always be reversed so that $p$ has the correct sign), it is possible to produce any $p$ required.

## References

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2. M. L. Kales, J. Appl. Phys. **24**, 604 (1953)
3. J. Proudman, *Dynamical oceanography*, Methuen, London, 1953, p. 220