DIFFRACTION OF ELASTIC WAVES BY A PENNY-SHAPED CRACK*

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Abstract. The diffraction of axisymmetric, harmonic elastic waves by a circular crack is considered. It is shown that the potential functions for the diffracted waves can be obtained from the solution of a pair of dual integral equations. The dual equations are transformed into integral equations of the second kind suitable for iteration at low frequencies. The principle of contraction mapping is used to discuss the convergence of the iteration scheme. The solution satisfies an edge condition.

1. The dual integral equations. The iterative solution to the problem of diffraction of elastic waves by a rigid circular disc was presented earlier [1]. In this paper the discussion is extended to the problem of the diffraction of axisymmetric harmonic waves by a circular crack of vanishing thickness, embedded in an otherwise homogeneous isotropic infinite elastic medium. Let (r, φ, z) denote the cylindrical polar coordinates of a point with reference to a system of axes having the origin at the centre of the crack and the z-axis perpendicular to its plane. Normalize all lengths with respect to the radius of the crack. Thus the crack is located at $z = 0$, $0 \leq r \leq 1$. For axisymmetric waves the incident as well as the total field are independent of φ. Then the displacement vector $\boldsymbol{u}(\mathbf{r}) \exp(-i\omega t)$ and the stress components $\tau_{pq}(\mathbf{r}) \exp(-i\omega t)$ in the diffracted field are

$$u_r(\mathbf{r}) = \frac{\partial}{\partial r}\left(\Phi + \frac{\partial \Psi}{\partial z}\right),$$
$$u_z(\mathbf{r}) = \frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Psi}{\partial z^2} + k_2^2 \Psi, \tag{1}$$

$$\tau_{rz}(\mathbf{r}) = \mu \frac{\partial}{\partial r}\left(2\frac{\partial \Phi}{\partial z} + 2\frac{\partial^2 \Psi}{\partial z^2} + k_2^2 \Psi\right),$$
$$\tau_{zz}(\mathbf{r}) = \mu\left\{2\frac{\partial}{\partial z}\left(\frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Psi}{\partial z^2} + k_2^2 \Psi\right) + (2k_1^2 - k_2^2)\Phi\right\}, \tag{2}$$
$$\tau_{rr}(\mathbf{r}) = \mu\left\{(2k_1^2 - k_2^2)\Phi + 2\frac{\partial^2}{\partial r^2}\left(\Phi + \frac{\partial \Psi}{\partial z}\right)\right\},$$

where the potential functions $\Phi(\mathbf{r}) \exp(-i\omega t)$ and $\Psi(\mathbf{r}) \exp(-i\omega t)$ are solutions of the wave equations

$$\nabla^2 \Phi + k_1^2 \Phi = 0, \qquad \nabla^2 \Psi + k_2^2 \Psi = 0, \tag{3}$$

μ is the rigidity of the medium and

$$k_1 = \omega/\alpha, \qquad k_2 = \omega/\beta, \tag{4}$$

α and β being the compressional and shear wave velocities in the material.

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Postulate solutions of (3) to be [1]

$$\Phi(\mathbf{r}) = \frac{1}{\omega^2}\int_0^\infty \left\{\mp P(\zeta) + \frac{\zeta}{\nu_1}Q'(\zeta)\right\}\exp(-\nu_1|z|)\zeta J_0(\zeta r)\,d\zeta,$$
$$\Psi(\mathbf{r}) = \frac{1}{\omega^2}\int_0^\infty \left\{\mp Q(\zeta) + \frac{\zeta}{\nu_2}P'(\zeta)\right\}\exp(-\nu_2|z|)J_0(\zeta r)\,d\zeta, \tag{5}$$

where

$$\nu_j^2 = \zeta^2 - k_j^2, \qquad \mathrm{Re}\,\nu_j > 0, \qquad j = 1, 2. \tag{6}$$

The $\mp$ sign in (5) refer to $z \gtrless 0$ respectively. Then for $z \neq 0$,

$$\tau_{rz} = -\frac{\mu}{\omega^2}\int_0^\infty \left\{2\zeta\nu_1\left(P \mp \frac{\zeta}{\nu_1}Q'\right)\exp(-\nu_1|z|) + (2\zeta^2 - k_2^2)\left(\mp Q + \frac{\zeta}{\nu_2}P'\right)\exp(-\nu_2|z|)\right\}\zeta J_1(\zeta r)\,d\zeta,$$
$$\tau_{zz} = \frac{\mu}{\omega^2}\int_0^\infty \left\{(2\zeta^2 - k_2^2)\left(\mp P + \frac{\zeta}{\nu_1}Q'\right)\exp(-\nu_1|z|) + 2\zeta\nu_2\left(Q \mp \frac{\zeta}{\nu_2}P'\right)\exp(-\nu_2|z|)\right\}\zeta J_0(\zeta r)\,d\zeta. \tag{7}$$

The boundary conditions on the crack are that the total normal stresses must vanish on both faces of the crack. If $\tau_{rz}^0\exp(-i\omega t)$, $\tau_{zz}^0\exp(-i\omega t)$ are the stresses in the incident waves, we must have, for $0 \le r < 1$,

$$\lim_{z\to-0}(\tau_{rz} + \tau_{rz}^0) = \lim_{z\to+0}(\tau_{rz} + \tau_{rz}^0) = 0,$$
$$\lim_{z\to-0}(\tau_{zz} + \tau_{zz}^0) = \lim_{z\to+0}(\tau_{zz} + \tau_{zz}^0) = 0. \tag{8}$$

Using (7), the above conditions give,

$$\int_0^\infty \{(2\zeta^2 - k_2^2)Q + 2\zeta^2 Q'\}\zeta J_1(\zeta r)\,d\zeta = 0,$$
$$\int_0^\infty \{(2\zeta^2 - k_2^2)P + 2\zeta^2 P'\}\zeta J_0(\zeta r)\,d\zeta = 0, \tag{9a}$$

$$\int_0^\infty \left\{2\nu_1 P + \frac{2\zeta^2 - k_2^2}{\nu_2}P'\right\}\zeta^2 J_1(\zeta r)\,d\zeta = +\omega^2 f_0(r), \tag{10a}$$

$$\int_0^\infty \left\{\frac{2\zeta^2 - k_2^2}{\nu_1}Q' + 2\nu_2 Q\right\}\zeta^2 J_0(\zeta r)\,d\zeta = -\omega^2 g_0(r), \tag{10b}$$

where $0 \le r < 1$ and

$$f_0(r) = \lim_{z\to0}\tau_{rz}^0(\mathbf{r})/\mu, \qquad g_0(r) = \lim_{z\to0}\tau_{zz}^0(\mathbf{r})/\mu. \tag{11}$$

Since there is no physical discontinuity in the properties of the medium outside the crack, all the components of the displacement and the stress must be continuous across $z = 0$, for $r > 1$. In particular, if u_r, u_z, τ_{rz} and τ_{zz} are to be continuous across $z = 0$, for $r > 1$, the unknown functions must satisfy the additional conditions

$$\int_0^\infty (P + P')\zeta^2 J_1(\zeta r)\, d\zeta = 0, \tag{12a}$$

$$\int_0^\infty (Q + Q')\zeta^2 J_0(\zeta r)\, d\zeta = 0, \tag{12b}$$

$$\begin{gathered} \int_0^\infty \{(2\zeta^2 - k_2^2)Q + 2\zeta^2 Q'\}\zeta J_1(\zeta r)\, d\zeta = 0, \\ \int_0^\infty \{(2\zeta^2 - k_2^2)P + 2\zeta^2 P'\}\zeta J_0(\zeta r)\, d\zeta = 0. \end{gathered} \tag{9b}$$

Combining (9a) and (9b) for all r,

$$(2\zeta^2 - k_2^2)P(\zeta) + 2\zeta^2 P'(\zeta) = 0, \qquad (2\zeta^2 - k_2^2)Q(\zeta) + 2\zeta^2 Q'(\zeta) = 0. \tag{13}$$

Substituting for P', Q' from (13) in (10) and (12),

$$\int_0^\infty \{4\zeta^2\nu_1\nu_2 - (2\zeta^2 - k_2^2)^2\} \frac{P(\zeta)}{\nu_2} J_1(\zeta r)\, d\zeta = 2\omega^2 f_0(r), \qquad 0 \le r < 1, \tag{14a}$$

$$\int_0^\infty P(\zeta) J_1(\zeta r)\, d\zeta = 0, \qquad r > 1, \tag{14b}$$

$$\int_0^\infty \{4\zeta^2\nu_1\nu_2 - (2\zeta^2 - k_2^2)^2\} \frac{Q(\zeta)}{\nu_1} J_0(\zeta r)\, d\zeta = -2\omega^2 g_0(r), \qquad 0 \le r < 1, \tag{15a}$$

$$\int_0^\infty Q(\zeta) J_0(\zeta r)\, d\zeta = 0, \qquad r > 1. \tag{15b}$$

It can be easily verified that (13), (14b) and (15b) are the necessary and sufficient conditions for the continuity of the displacement and the stresses across $z = 0$, outside the area occupied by the crack. Rewrite the above dual integral equations for $P(\zeta)$ and $Q(\zeta)$ as

$$\begin{gathered} \int_0^\infty \zeta P(\zeta) J_1(\zeta r)\, d\zeta = \omega^2 f_0(r)/(k_2^2 - k_1^2) + f_1(r), \qquad 0 \le r < 1, \\ \int_0^\infty P(\zeta) J_1(\zeta r)\, d\zeta = 0, \qquad r > 1 \end{gathered} \tag{16}$$

and

$$\begin{gathered} \int_0^\infty \zeta Q(\zeta) J_0(\zeta r)\, d\zeta = -\omega^2 g_0(r)/(k_2^2 - k_1^2) + g_1(r), \qquad 0 \le r < 1, \\ \int_0^\infty Q(\zeta) J_0(\zeta r)\, d\zeta = 0, \qquad r > 1, \end{gathered} \tag{17}$$

where

$$f_1(r) = \int_0^\infty F(\zeta) P(\zeta) J_1(\zeta r)\, d\zeta, \tag{18a}$$

$$F(\zeta) = \zeta - \{4\zeta^2\nu_1\nu_2 - (2\zeta^2 - k_2^2)^2\}/2\nu_2(k_2^2 - k_1^2), \tag{18b}$$

$$g_1(r) = \int_0^\infty G(\zeta)Q(\zeta)J_0(\zeta r)\, d\zeta, \tag{19a}$$

$$G(\zeta) = \zeta - \{4\zeta^2\nu_1\nu_2 - (2\zeta^2 - k_2^2)^2\}/2\nu_1(k_2^2 - k_1^2). \tag{19b}$$

Note that $\omega^2 f_0(r)/(k_1^2 - k_2^2)$ and $\omega^2 g_0(r)/(k_1^2 - k_2^2)$ remain finite for $\omega \to 0$ and as $\zeta \to \infty$

$$F(\zeta) \sim O\left\{\frac{k_1^4 + k_2^4}{8(k_2^2 - k_1^2)}\, \zeta^{-1}\right\}, \tag{20a}$$

$$G(\zeta) \sim O\left\{\frac{3k_1^4 - 4k_1^2k_2^2 + 3k_2^4}{8(k_2^2 - k_1^2)}\, \zeta^{-1}\right\}. \tag{20b}$$

Also, $F(\zeta)$ and $G(\zeta)$ are finite as $\zeta \to 0$. The foregoing observations will be relevant when the integral equations are solved by iteration.

If the right-hand sides of (16) and (17) are considered as known quantities for the moment, a formal solution for each of the dual integral equations can be written down. Following Sneddon [2],

$$P(\zeta) = \frac{2}{\pi\zeta} \int_0^1 (\sin \eta\zeta - \eta\zeta \cos \eta\zeta)\, d\eta \int_0^1 \frac{t^2 A(\eta t)}{(1 - t^2)^{1/2}}\, dt, \tag{21}$$

$$Q(\zeta) = \frac{2}{\pi} \int_0^1 \eta \sin \eta\zeta\, d\eta \int_0^1 \frac{tB(\eta t)}{(1 - t^2)^{1/2}}\, dt, \tag{22}$$

where

$$A(r) = +\frac{\omega^2 f_0(r)}{k_2^2 - k_1^2} + f_1(r), \qquad B(r) = -\frac{\omega^2 g_0(r)}{k_2^2 - k_1^2} + g_1(r). \tag{23}$$

After integration by parts and rearrangement,

$$P(\zeta) = P_0(\zeta) + \frac{1}{\pi} \int_0^\infty t^{-1}P(t)F(t)\left\{\frac{\sin (t - \zeta)}{t - \zeta} + \frac{\sin (t + \zeta)}{t + \zeta} - \frac{2 \sin t \sin \zeta}{t\zeta}\right\} dt, \tag{24}$$

$$Q(\zeta) = Q_0(\zeta) + \frac{1}{\pi} \int_0^\infty t^{-1}Q(t)G(t)\left\{\frac{\sin (t - \zeta)}{t - \zeta} - \frac{\sin (t + \zeta)}{t + \zeta}\right\} dt, \tag{25}$$

where

$$P_0(\zeta) = \frac{2\omega^2}{\pi(k_2^2 - k_1^2)\zeta} \int_0^1 (\sin \eta\zeta - \eta\zeta \cos \eta\zeta)\, d\eta \int_0^1 \frac{f_0(\eta t)t^2}{(1 - t^2)^{1/2}}\, dt, \tag{26}$$

$$Q_0(\zeta) = -\frac{2\omega^2}{\pi(k_2^2 - k_1^2)} \int_0^1 \eta \sin \eta\zeta\, d\eta \int_0^1 \frac{g_0(\eta t)t}{(1 - t^2)^{1/2}}\, dt. \tag{27}$$

Equations (24) and (25) can also be written conveniently as follows

$$P(\zeta) = P_0(\zeta) + \int_0^1 R_P(\eta) \cos \eta\zeta\, d\eta, \tag{28}$$

$$Q(\zeta) = Q_0(\zeta) + \int_0^1 S_Q(\eta) \sin \eta\zeta\, d\eta, \tag{29}$$

where

$$R_P(\eta) = \frac{2}{\pi} \int_0^\infty t^{-1} F(t) P(t) (\cos \eta t - t^{-1} \sin t) \, dt,$$

$$S_Q(\eta) = \frac{2}{\pi} \int_0^\infty t^{-1} G(t) Q(t) \sin \eta t \, dt,$$

with

$$\int_0^1 R_P(\eta) \, d\eta = 0.$$

In (26)

$$\int_0^1 \frac{t^2 f_0(\eta t)}{(1 - t^2)^{1/2}} \, dt = \frac{d}{d\eta} \left\{ \eta \int_0^1 (1 - t^2)^{1/2} f_0(\eta t) \, dt \right\}$$

so that

$$P_0(\zeta) = \int_0^1 R_0(\eta) \cos \eta\zeta \, d\eta, \tag{30}$$

where

$$R_0(\eta) = \frac{2\omega^2}{\pi(k_2^2 - k_1^2)} \left[\int_0^1 (1 - t^2)^{1/2} f_0(t) \, dt - \frac{d}{d\eta} \left\{ \eta^2 \int_0^1 (1 - t^2)^{1/2} f_0(\eta t) \, dt \right\} \right]. \tag{31}$$

It is to be noted that $R_0(\eta)$ satisfies

$$\int_0^1 R_0(\eta) \, d\eta = 0. \tag{32}$$

From (27),

$$Q_0(\zeta) = \int_0^1 S_0(\eta) \sin \eta\zeta \, d\eta, \tag{33}$$

where

$$S_0(\eta) = -\frac{2\omega^2 \eta}{\pi(k_2^2 - k_1^2)} \int_0^1 \frac{t^2 g_0(\eta t)}{(1 - t^2)^{1/2}} \, dt. \tag{34}$$

2. Solution of the integral equations. Solutions to the integral equations (24), (25) will now be obtained under the assumption that $P(\zeta)$, $Q(\zeta)$ satisfy the following conditions:

(i) $P(\zeta)$, $Q(\zeta)$, $\zeta P(\zeta)$, $\zeta Q(\zeta)$ are bounded, continuous in $[0, \infty[$

(ii) $$P(\zeta) = \int_0^1 p(t) \cos \zeta t \, dt, \qquad Q(\zeta) = \int_0^1 q(t) \sin \zeta t \, dt, \tag{35}$$

where $p(t)$, $q(t)$ are bounded in $[0, 1]$ and differentiable in $0 < t < 1$. In addition, $p(t)$ satisfies

$$\int_0^1 p(t) \, dt = 0. \tag{36}$$

Before proceeding to solve the integral equations, however, it must be shown that the above assumptions are consistent with the formal analysis that led to these equations.

For $z \neq 0$, the integral expressions for the displacements and the stress components are convergent due to the choice of the branches in (6). However, in applying the boundary conditions in the plane of the crack the orders of the ζ-integrals have been interchanged with the limiting process $z \to 0$. This interchange is justified if the following integrals are convergent,

$$I_1(r) = \int_0^\infty P(\zeta)J_1(\zeta r)\,d\zeta, \qquad I_2(r) = \int_0^\infty Q(\zeta)J_0(\zeta r)\,d\zeta,$$

$$I_3(r) = \int_0^\infty \zeta P(\zeta)J_1(\zeta r)\,d\zeta, \qquad I_4(r) = \int_0^\infty \zeta Q(\zeta)J_0(\zeta r)\,d\zeta,$$

$$I_5(r) = \int_0^\infty F(\zeta)P(\zeta)J_1(r\zeta)\,d\zeta, \qquad I_6(r) = \int_0^\infty G(\zeta)Q(\zeta)J_0(\zeta r)\,d\zeta.$$

Using (35), (36)

$$\begin{aligned} I_1(r) &= \frac{1}{r}\int_r^1 p(t)\left(1 - \frac{t}{(t^2 - r^2)^{1/2}}\right) dt, \qquad 0 < r < 1, \\ &= 0, \qquad r > 1, \end{aligned} \tag{37}$$

$$\begin{aligned} I_2(r) &= \int_r^1 \frac{q(t)}{(t^2 - r^2)^{1/2}}\,dt, \qquad 0 < r < 1, \\ &= 0, \qquad r > 1, \end{aligned} \tag{38}$$

$$\begin{aligned} I_3(r) &= \frac{p(1)}{r(r^2 - 1)^{1/2}} - \frac{1}{r}\int_0^1 \frac{tp'(t)}{(r^2 - t^2)^{1/2}}\,dt, \qquad r > 1, \\ &= -\int_0^r \frac{tp'(t)}{(r^2 - t^2)^{1/2}}\,dt, \qquad 0 < r < 1, \end{aligned} \tag{39}$$

$$\begin{aligned} I_4(r) &= \frac{q(0)}{r} - \frac{q(1)}{(r^2 - 1)^{1/2}} + \int_0^1 \frac{q'(t)\,dt}{(r^2 - t^2)^{1/2}}\,dt, \qquad r > 1, \\ &= \frac{q(0)}{r} + \int_0^r \frac{q'(t)}{(r^2 - t^2)^{1/2}}\,dt, \qquad 0 < r < 1. \end{aligned} \tag{40}$$

The integrals I_5, I_6 are absolutely integrable because of (20).

The principle of contraction mapping can now be used to solve the integral equations (24) and (25). Consider Equation (24) first. Let E be the space of bounded, continuous functions $f(\zeta)$ on $[0, \infty[$, such that $\zeta f(\zeta)$ is bounded. Let E be normed by

$$||f|| = \sup_{\zeta>0} |f(\zeta)| + \sup_{\zeta>0} |\zeta f(\zeta)|.$$

Then E is a Banach space. Let T_1 be the linear operator defined on E as follows:

$$(T_1 f)(\zeta) = P_0(\zeta) + \int_0^1 R_f(\eta) \cos \eta\zeta\,d\eta. \tag{41}$$

It can be shown that T_1 takes E into E. Furthermore, for any two f, g in E,

$$||T_1 f - T_1 g|| = \sup_{\zeta>0} \left| \int_0^1 \{R_f(\eta) - R_g(\eta)\} \cos \eta\zeta\,d\eta \right| + \sup_{\zeta>0} \left| \zeta \int_0^1 \{R_f(\eta) - R_g(\eta)\} \cos \eta\zeta\,d\eta \right|$$

$$\leq \alpha_1 \, ||f - g||,$$

where

$$\alpha_1 = \max_{0<t<\infty} \{t^{-1}(1 - t^{-1} \sin t),\, t^{-2}(1 - t^{-1} \sin t)\} \cdot \frac{2}{\pi} \int_0^\infty |F(t)| \, dt. \tag{42}$$

Thus, if $\alpha_1 < 1$, T_1 is a contraction and equation (24) can be solved by successive approximations. If we take the first approximation to be equal to $P_0(\zeta)$, the error after the nth iteration is ϵ_n,

$$|\epsilon_n| < \frac{\alpha_1^n}{1 - \alpha_1} \, ||P_0||. \tag{43}$$

The solution $P(\zeta)$ obtained by the successive approximations can be shown to satisfy conditions (i) and (ii) above because of (28), (30).

Now consider Equation (25). Let T_2 be the operator defined on E as follows:

$$(T_2 f)(\zeta) = Q_0(\zeta) + \int_0^1 S_f(\eta) \sin \eta\zeta \, d\eta. \tag{44}$$

Then T_2 takes E into E. Furthermore, for any two f, g in E,

$$\begin{aligned} ||T_2 f - T_2 g|| &= \sup_{\zeta>0} \left| \int_0^1 \{S_f(\eta) - S_g(\eta)\} \sin \eta\zeta \, d\eta \right| \\ &\qquad + \sup_{\zeta>0} \left| \zeta \int_0^1 \{S_f(\eta) - S_g(\eta)\} \sin \eta\zeta \, d\eta \right| \\ &\leq \frac{2}{\pi} \int_0^\infty |t^{-1}(1 - t^{-1} \sin t)| \, |tQ(t)| \, |G(t)| \, dt \\ &\qquad + \frac{6}{\pi} \int_0^\infty |Q(t)| \, |G(t)| \, dt \\ &< \alpha_2 \, ||f - g||, \end{aligned}$$

where

$$\alpha_2 = \frac{6}{\pi} \int_0^\infty |G(t)| \, dt. \tag{45}$$

If $Q_0(\zeta)$ is taken to be equal to the first order approximation the error involved after the nth iteration is δ_n,

$$|\delta_n| < \frac{\alpha_2^n}{1 - \alpha_2} \, ||Q_0||.$$

Again it can be verified that the solution $Q(\zeta)$ obtained above satisfies conditions (i) and (ii).

3. Edge condition. It can be seen from Equations (37)–(40) that the displacement and the stress functions are singular at points on the edge of the crack. In order that the solution obtained above be physically plausible, it must be verified that the above singularities do not give rise to real forces at the edge.

It is proved below that if $P(\zeta)$ and $Q(\zeta)$ satisfy conditions (i) and (ii) given above, no net body forces are induced on the edge. Enclose the edge of the crack by a surface S_ϵ formed by the concentric cylinders $r = 1 \pm \epsilon$ and the parallel planes $z = \pm\epsilon$ ($\epsilon > 0$), and compute the integral

$$F_i = \int_{S_\epsilon} \tau_{ij} n_j \, dS, \qquad i, j = r, z,$$

n_i being the outward unit normal to S_ϵ. Then

$$F_z = -2\pi \int_{-\epsilon}^{\epsilon} (r\tau_{rz})_{r=1-\epsilon}\, dz + 2\pi \int_{-\epsilon}^{\epsilon} (r\tau_{rz})_{r=1+\epsilon}\, dz$$

$$- 2\pi \int_{1-\epsilon}^{1+\epsilon} (\tau_{zz})_{z=-\epsilon} r\, dr + 2\pi \int_{1-\epsilon}^{1+\epsilon} (\tau_{zz})_{z=\epsilon} r\, dr,$$

$$= \frac{4\pi\mu}{\omega^2} \int_0^\infty \left\{\frac{2\zeta^2}{\nu_1}(1 - \exp(-\nu_1\epsilon)) - \frac{2\zeta^2 - k_2^2}{\nu_2}(1 - \exp(-\nu_2\epsilon))\right\} \zeta Q(\zeta)[rJ_1(\zeta r)]_{1-\epsilon}^{1+\epsilon}\, d\zeta$$

$$- \frac{4\pi\mu}{\omega^2} \int_{1-\epsilon}^{1+\epsilon} r\, dr \int_0^\infty \{(2\zeta^2 - k_2^2)\exp(-\nu_1\epsilon) - 2\zeta^2 \exp(-\nu_2\epsilon)\}\zeta P(\zeta) J_0(\zeta r)\, d\zeta.$$

In the limit $\epsilon \to 0$, the first integral in the right-hand side of the above equation vanishes and the second integral becomes

$$4\pi\mu/\beta^2 \lim_{\epsilon \to 0} \int_{1-\epsilon}^{1+\epsilon} rI_1(r)\, dr,$$

where $I_1(r)$ is given by (37). Thus,

$$\lim_{\epsilon\to 0} F_z = \frac{4\pi\mu}{\beta^2} \lim_{\epsilon\to 0} \int_{1-\epsilon}^{1} dr \int_r^1 p(t)\left\{1 - \frac{t}{(t^2 - r^2)^{1/2}}\right\} dt.$$

Since r, t are both less than 1, and $p(t)$ is bounded in [0, 1],

$$|F_z| < \frac{4\pi\mu m}{\beta^2} \int_{1-\epsilon}^{1} (1 - r + (1 - r^2)^{1/2})\, dr \to 0 \quad \text{as} \quad \epsilon \to 0,$$

m being the upper bound of $p(t)$ in [0, 1]. The radial component of force F_r can be treated similarly.

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