# ON NONLINEAR STEADY-STATE SOLUTIONS TO MOVING LOAD PROBLEMS[1]

BY

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**Abstract.** In certain moving load problems involving elastic solids or structures with an idealized infinite or semi-infinite spatial domain, steady-state solutions of the governing equations of motion are frequently sought as an alternative to a more complex transient analysis. In seeking a solution of this type the implication is, of course, that it represents the limit of a transient problem. While this appears to be generally true of well-posed linear elastic problems, the same cannot be said of nonlinear elastic formulations. To substantiate this point the case of an infinite-length string supported by a nonlinear elastic foundation and subjected to a concentrated load moving with constant velocity is studied in this paper. It is shown that although the steady-state solution of the problem is unique for load velocities both above and below the signal velocity of the string, that above is locally unstable and therefore cannot represent the limit of transient motions.

**1. Introduction.** The response of structures to moving loads has received the attention of a substantial number of investigators in recent years. In many instances such problems are cast in the mathematically idealized form of a load moving rectilinearly with constant velocity on a system of infinite extension in the direction of the load trajectory. As a further idealization, and in lieu of a formidable transient analysis, one often seeks a steady-state solution of the governing equations of motion, i.e., a deformation pattern appearing as time-invariant to an observer moving with the load. It is usually assumed, without later verification, that such solutions represent the limiting case of a transient problem in which the load is applied and brought up to speed in some manner. If this assumption is true, a steady-state analysis can convey considerable knowledge about the problem in question. However, if false, this assumption can be quite misleading.

One prerequisite of a physically meaningful steady-state solution is that it be stable with respect to small imposed disturbances; it is clear that an unstable steady-state wave form cannot be expected as the limit of a transient motion. It is interesting to note that while well-posed linear elastic problems appear generally to satisfy this requirement, the same cannot be said of nonlinear elastic problems. Seemingly well-conditioned nonlinear cases can be constructed for which the steady-state solution is unique but unstable in certain load-velocity regions. In particular, this phenomena occurs in certain one-dimensional elastic cases (viz., involving strings, beams, axisymmetric motion of cylindrical shells, etc.) when such items as dissipation are omitted.

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In this paper the above remarks concerning stability are illustrated by an elementary nonlinear example, namely a string of infinite length supported by a nonlinear foundation and subjected to a concentrated load moving with constant velocity. In particular, it is shown that while the associated steady-state solution is unique for load velocities both above and below the signal velocity of the string, the solution above the signal velocity is locally unstable and therefore does not represent the limit of transient motions.

**2. A nonlinear wave equation.** Consider the wave equation

$$y_{xx} - y_{tt} + f(x, t) - g(y) = 0. \tag{1}$$

This is the equation governing the small transverse displacements of a string or flexible cable under constant tension, with uniform mass per unit length, carrying a distributed load $f(x, t)$ and supported by or imbedded in a nonlinear elastic foundation whose restoring force is $g(y)$.

Let us, as the objective of this discussion, investigate the stability of solutions of (1) that are defined on the domain $x \in (-\infty, \infty)$, $t \in (-\infty, \infty)$ and for which

(i) $f(x, t) = P\,\delta(x - vt)$, where $\delta(\ )$ denotes the Dirac delta function and $P$, $v$ are positive constants.

(ii) $y(x, t) = y(x - vt)$ where $y$ is a bounded, continuous function of $x - vt$.

(iii) $g(y)$ is such that $g(0) = 0$, $g'(y) \geq \delta > 0$ and, except for the linear case $g(y) = ky$, $k > 0$, either (a) $g(y)$ is a smooth function of $y$ with $g'(y)$ strictly decreasing with increasing $|y|$ or (b) $g(y) \in C^n$, where $n$ is finite and $n > 1$, but $g(y)$ is not infinitely differentiable at $y = 0$.

According to the domain of definition the string is of infinite length. The form (i) of $f(x, t)$ is representative of a concentrated load moving in the positive $x$ direction with a constant velocity, $v$. The solutions (ii) are of the steady-state form mentioned previously. The restrictions (iii) on $g(y)$ will allow us to demonstrate certain instability conditions without a complex and involved analysis.

**3. Steady-state solution.** For future discussion, it will be convenient to introduce the following new variables

$$\xi = x - vt, \qquad \tau = t. \tag{2}$$

Under (2) Eq. (1) becomes

$$(1 - v^2)y_{\xi\xi} + 2vy_{\xi\tau} - y_{\tau\tau} - g(y) + P\,\delta(\xi) = 0. \tag{3}$$

According to the above Galilean transformation, all motions are now referred to a coordinate system moving with the load.

In terms of (3) a steady-state solution of (1) is a bounded, continuous solution of (3) satisfying $y_\tau = 0$, i.e., a solution of

$$(1 - v^2)y_s''(\xi) - g(y_s) + P\,\delta(\xi) = 0. \tag{4}$$

Here the subscript "$s$" refers to "steady-state" and $(\ )'$ denotes $d(\ )/d\xi$.

As is evident from (4), the character of $y_s(\xi)$ can be expected to differ markedly in the regions $v < 1$ and $v > 1$. The case $v = 1$ corresponds to a load traveling with the characteristic or signal velocity of the string. Thus $v < 1$ can be envisioned as being "subsonic" and $v > 1$ as "supersonic". It will be expedient to treat each region separately (the case $v = 1$ is of course singular; solutions of the form (ii) do not exist for $v = 1$).

*The Case* $v < 1$. For $v < 1$ the desired solution of (4) is the bounded solution of

$$(1 - v^2)y_s''(\xi) - g(y_s) = 0 \tag{5a}$$

in the intervals $\xi \in (-\infty, 0)$ and $\xi \in (0, \infty)$ satisfying the jump (resulting from the delta function) and continuity conditions

$$y_s'(0^+) - y_s'(0^-) = -P/(1 - v^2), \tag{5b}$$

$$y_s(0^+) - y_s(0^-) = 0. \tag{5c}$$

The first two integrals of (5a) fulfilling these requirements are

$$y_s' = \pm\left[\frac{2}{1 - v^2} \int_0^{y_s} g(\lambda)\, d\lambda\right]^{1/2}, \tag{6a}$$

$$\int_0^{y_s} \frac{d\lambda}{\left[\frac{1}{1 - v^2} \int_0^{\lambda} g(\mu)\, d\mu\right]^{1/2}} = \pm[\xi + \text{const.}]. \tag{6b}$$

The positive and negative signs in Eqs. (6) apply for $\xi < 0$ and $\xi > 0$ respectively. We note that the inequality

$$\int_0^{y_s} g(\lambda)\, d\lambda > 0, \qquad |y_s| > 0 \tag{6c}$$

is satisfied since $g'(y) > 0$, $g(0) = 0$.

Equations (5) and (6) imply that $y_s(\xi)$ for $v < 1$ is a symmetric, positive function of $\xi$, possessing a maximum at $\xi = 0$ and strictly decreasing to zero on either side of $\xi = 0$ as $\xi \to \pm\infty$.

The steady-state solution is unique for $v < 1$ in the sense that one and only one function $y_s(\xi)$ is defined for a given value of $P$. This can be observed as follows: from Eqs. (5), (6a) and the symmetry property $y_s(\xi) = y_s(-\xi)$, one deduces that

$$dP/dy_0 = 4g(y_0)/P. \tag{7}$$

Here $y_0$ denotes $y_s(0)$. Since $y_s(\xi) > 0$ for $P > 0$, which implies $g(y_s) > 0$, one concludes from (7) that $dP/dy_0$ is of one sign and nonzero for $P > 0$. Thus, each value of $P$ defines a unique value of $y_0$ (we require that $y_0 = 0$ when $P = 0$). This indicates that each value of $P$ defines a unique set of initial values $y_s(0^+)$, $y_s'(0^+)$ for the interval $\xi \in (0, \infty)$ and another unique set, $y_s(0^-)$, $y_s'(0^-)$ for the interval $\xi \in (-\infty, 0)$. Thus, in each interval the problem can be cast as an initial value problem. Under the assumption that, in addition to being continuous, $g(y)$ satisfies at least a Lipschitz condition, one can now apply the theorem of Cauchy–Lipschitz to deduce the uniqueness of solutions of (5a) with the matching conditions (5b, c).

A remark concerning $g(y)$ other than that for which $g'(y) > 0$ is appropriate at this point. We note that steady-state solutions of the form (6) exists if $\int_0^{y_s} g(\mu)\, d\mu > 0$. Equation (7) indicates such solutions have the property that $y_s(0)$ is a monotonically increasing function of $P$ when $g(y_s) > 0$. Thus, although $g(y)$ may exhibit a "locally unstable" character, as is illustrated in Fig. 1, the deflection directly under the load $P$ increases monotonically with $P$, and the system as a whole appears well behaved.

*Example* $v < 1$. As an illustration of what one might expect for $v < 1$, suppose $g(y)$ is defined by

$$g(y) = ay + by^3; \qquad a > 0, \quad b < 0 \tag{8}$$

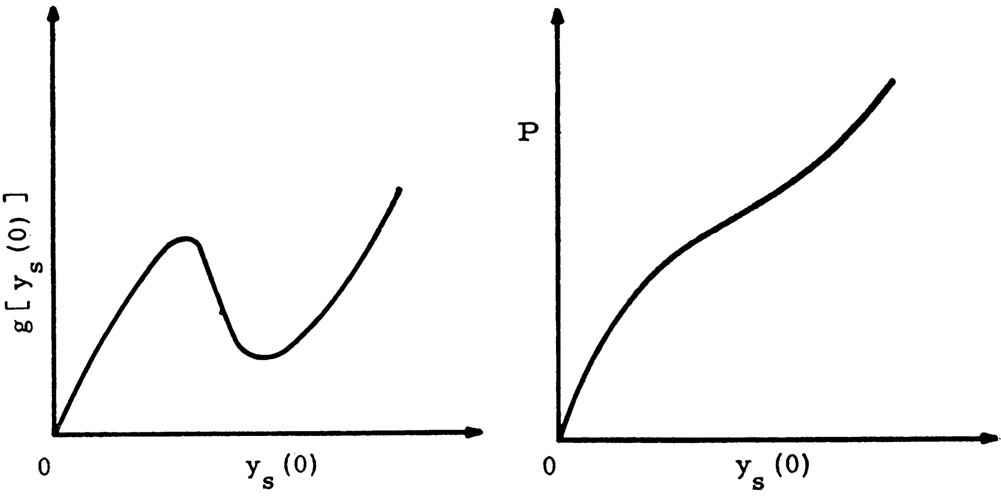


Fig. 1. Deflection under load $P$ for a "locally unstable" foundation; $v < 1$.

and let us investigate the solution $y_s(\xi)$ in the range of $|y_s|$ for which $g'(y) > 0$ (i.e., for $|y_s| < |a/3b|$).

The bounded, continuous solution of (5a) subject to the foundation (8) and the jump relation (5b) is (verify by substitution)

$$y_s(\xi) = \frac{2e^{-|\psi|}}{[\gamma - (\lambda/\gamma)e^{-2|\psi|}]} \tag{9a}$$

where

$$\begin{aligned} \psi &= [a/(1 - v^2)]^{1/2}(\xi), \\ \lambda &= b/2a, \\ \gamma &= (1 + [1 + \lambda y_0^2]^{1/2})/y_0 , \\ y_0 \equiv y_s(0) &= \frac{\bar{P}}{2^{1/2}} [1 + (1 + \lambda \bar{P}^2)^{1/2}]^{-1/2}, \\ \bar{P} &= P[a(1 - v^2)]^{-1/2}. \end{aligned} \tag{9b}$$

It can be demonstrated that the denominator of (9a) does not vanish for $P < \infty$.

If $b = 0$, Eq. (9a) reduces to the classical linear result

$$y_s(\xi) = \frac{\bar{P}}{2} \exp(-|\psi|); \qquad b = 0. \tag{9c}$$

*The Case* $v > 1$. Consider now the case $v > 1$. Since Eq. (1) is hyperbolic with unit characteristic speed, the function $y_s(\xi)$ will be required to vanish for $\xi \geq 0$ when $v > 1$. In lieu of the delta function and the continuity requirements mentioned previously, one obtains $y_s(\xi)$ for $\xi < 0$ when $v > 1$ as the solution of

$$(v^2 - 1)y_s'' + g(y_s) = 0, \quad \xi < 0 \tag{10a}$$

subject to the initial data

$$y_s(0^-) = 0, \quad y_s'(0^-) = -P/(v^2 - 1). \tag{10b}$$

A basic difference in the character of $y_s$ in the "subsonic" and "supersonic" regions is now evident. Equation (10a) indicates $y_s$ is a *periodic* function of $\xi$ for $v > 1$ and $\xi < 0$. Further $y_s$ vanishes for $\xi \geq 0$. This is in contrast to the *symmetric, monotone* behavior of $y_s$ for $v < 1$. A comparison of the two cases is illustrated schematically in Fig. 2.

Since the load $P$ enters the problem only through the initial data, one concludes, again by application of the theorem of Cauchy–Lipschitz, that $y_s(\xi)$ is unique for a given value of $P$ in the region $v > 1$ if $g(y)$ satisfies a Lipschitz condition.

*Example* $v > 1$. Let us again assume $g(y)$ is defined by Eq. (8). Then $y_s(\xi)$ for $v > 1$ is readily obtained as

$$\begin{aligned} y_s(\xi) &= A \operatorname{sn}\left(-\frac{P^*\Omega}{A} \,\middle|\, \frac{\alpha A^2}{1 - \alpha A^2}\right), \qquad \xi < 0, \\ &= 0, \qquad \xi \geq 0, \end{aligned} \tag{11a}$$

where

$$\begin{aligned} \Omega &= [a/(v^2 - 1)]^{1/2}(\xi), \qquad \alpha = b/a, \\ A &= -2^{1/2}(P^*)(1 + (1 - 2\alpha P^{*2})^{1/2})^{-1/2}, \qquad P^* = (P)[a(v^2 - 1)]^{-1/2}. \end{aligned} \tag{11b}$$

The quantity sn $(u \mid m)$ in (11) in the Jacobian elliptic sine function with argument $u$ and parameter $m$. The notation is that of [1].

When $b = 0$ Eq. (11a) reduces to the classical linear result

$$\begin{aligned} y_s(\xi) &= -P^* \sin \Omega, \qquad \xi < 0 \\ &= 0, \qquad\qquad \xi \geq 0 \end{aligned} ; \qquad b = 0. \tag{12}$$

**4. Stability.** We now turn to the stability of $y_s(\xi)$. Accordingly, a disturbance $\eta(\xi, \tau)$ is superposed upon $y_s(\xi)$ in such a way that the total motion $y(\xi, \tau)$ is given by

$$y(\xi, \tau) = y_s(\xi) + \eta(\xi, \tau). \tag{13}$$

Combining (13) and (3) and, within the context of a local stability analysis, retaining only those terms of $O(\eta)$, one obtains the following variational equation governing the disturbance $\eta$:

$$(1 - v^2)\eta_{\xi\xi} + 2v\eta_{\xi\tau} - \eta_{\tau\tau} - g'(y_s)\eta = 0. \tag{14}$$

In terms of (14) the stability problem can be stated as follows: to determine the stability of the "undisturbed" state $\eta \equiv 0$, with respect to a set $\mathfrak{F}$ of "disturbed" motions which are solutions of (14). For purposes of this study the set $\mathfrak{F}$ will consist of all $C^2$ solutions of (14) with compact support, i.e., $C^2$ solutions vanishing outside a finite, closed $\xi$ interval for any instant $\tau$.

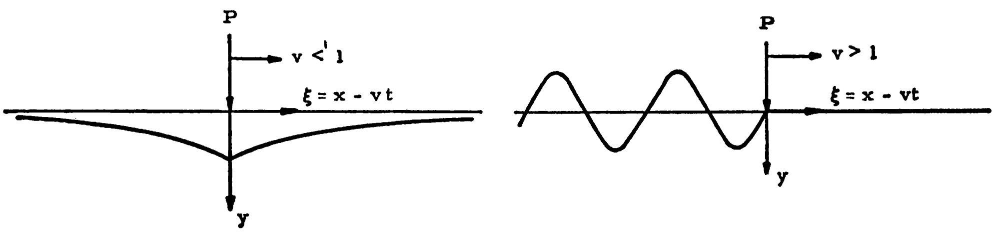


Fig. 2. Steady-state configurations for $v < 1$ and $v > 1$.

*The Case* $v < 1$. For $v < 1$, $y_s(\xi)$ is stable for all $P < \infty$ in the following sense: for any $\epsilon > 0$ we have $\sup_\xi |\eta| < \epsilon$ for all $\tau \geq 0$ provided $[(2)E(0)/\inf g'(y)]^{1/2} < \epsilon$. Here $E(\tau)$ is defined as

$$E(\tau) = \int_{-\infty}^{\infty} \{(1 - v^2)\eta_\xi^2 + \eta_\tau^2 + g'(y_s)\eta^2\} \, d\xi. \tag{15}$$

In other words, disturbances of the class $\mathfrak{F}$ can be maintained arbitrarily small for all time by requiring that the "energy" associated with the initial disturbance be made sufficiently small.

The foregoing can be verified as follows: by virtue of the variational equation (14) and the assumption $\eta \in \mathfrak{F}$, $E(\tau)$ satisfies

$$\frac{dE(\tau)}{d\tau} = 2v \int_{-\infty}^{\infty} \eta_{\xi\tau}\eta_\tau \, d\xi = v[\eta_\tau^2]_{\xi=-\infty}^{\infty} = 0. \tag{16}$$

Thus $E(\tau) = E(0) =$ constant. Now,

$$\eta^2 = \int_{-\infty}^{\xi} \frac{\partial}{\partial \xi}(\eta^2) \, d\xi = 2 \int_{-\infty}^{\xi} \eta\eta_\xi \, d\xi \tag{17}$$

and, applying the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} \eta^2 &\leq 2\left[\int_{-\infty}^{\xi} \eta^2 \, d\xi\right]^{1/2}\left[\int_{-\infty}^{\xi} \eta_\xi^2 \, d\xi\right]^{1/2} \\ &\leq 2\left[\int_{-\infty}^{\infty} \eta^2 \, d\xi\right]^{1/2}\left[\int_{-\infty}^{\infty} \eta_\xi^2 \, d\xi\right]^{1/2} \\ &\leq \frac{(2)E(0)}{\inf g'(y)} \end{aligned} \tag{18}$$

or, since the right-hand side is independent of $\tau$,

$$\sup{}_\xi |\eta| \leq \left[\frac{(2)E(0)}{\inf g'(y)}\right]^{1/2}.$$

*The Case* $v > 1$. For $v > 1$ we have the somewhat surprising result: if $g(y)$ is a *nonlinear* function of $y$, $y_s(\xi)$ is *unstable* for all $P$ and $v$ in the sense that the variational equation possesses solutions $\eta \in \mathfrak{F}$ which grow exponentially in time, i.e., $\sup_x |\eta(x, t)| > c_1 \exp(c_2 t)$ where $c_1, c_2 > 0$. In view of this, the steady-state solution cannot be expected as the limit of a transient motion for $v > 1$.

To substantiate the above statement consider separated solutions of the variational equation of the form

$$\eta(\xi, \tau) = \exp(i\sigma\tau) \, f(\xi) \tag{19}$$

where $\sigma$ is a real, positive constant. The function $f(\xi)$ satisfies the equation

$$(v^2 - 1)f'' - 2v\sigma i f' + [-\sigma^2 + g'(y_s)]f = 0. \tag{20}$$

By the substitution

$$f(\xi) = u(\xi) \exp[i\sigma v\xi/(v^2 - 1)], \tag{21}$$

Eq. (20) reduces to

$$u'' + [g'(y_s)/(v^2 - 1) + \sigma^2/(v^2 - 1)^2]u = 0. \tag{22}$$

Since $y_s(\xi) = 0$ for $\xi \geq 0$ when $v > 1$, $g'(y_s) = g'(0) =$ constant for $\xi \geq 0$ and Eq. (22) has the solution

$$u(\xi) = B_1 \sin K\xi + B_2 \cos K\xi, \qquad \xi \geq 0 \tag{23a}$$

where $B_1$ and $B_2$ are constants of integration and

$$K = g'(0)/(v^2 - 1) + \sigma^2/(v^2 - 1)^2. \tag{23b}$$

If we assume $g(y)$ is a nonlinear function of $y$, then $g'(y_s)$ is a periodic function of $\xi$ for $\xi < 0$ and can be written

$$g'(y_s)/(v^2 - 1) = \Lambda(P, v) + q(\xi; P, v) \tag{24}$$

where $\Lambda$ ($>0$) represents the average value of $g'(y_s)/(v^2 - 1)$ over one period, i.e., $q$ is assumed to be periodic with zero average value over one period.

Substitution of (24) into (22) yields the Hill equation

$$u'' + [\lambda(P, v, \sigma) + q(\xi; P, v)]u = 0, \qquad \xi < 0 \tag{25}$$

where

$$\lambda(P, v, \sigma) = \Lambda(P, v) + \sigma^2/(v^2 - 1)^2. \tag{26}$$

According to Floquet's theory, equation (25) possesses solutions of the form

$$\begin{aligned} u(\xi) &= B_3 \exp(\nu_1\xi)\varphi_1(\xi) + B_4 \exp(\nu_1\xi)\varphi_2(\xi) && \text{if } \nu_1 \neq \nu_2, \\ &= B_3 \exp(\nu\xi)\varphi_1(\xi) + B_4 \exp(\nu\xi)\left[\frac{a\xi}{\Omega}\varphi_1(\xi) + \varphi_3(\xi)\right] && \text{if } \nu_1 = \nu_2 = \nu. \end{aligned} \quad ; \ \xi < 0 \tag{27}$$

Here $B_3$, $B_4$, $a$, are constants, $\varphi_1(\xi)$, $\varphi_2(\xi)$ and $\varphi_3(\xi)$ are periodic functions with period $\Omega$ and $\nu_i$ are, in general, complex valued. We note that sufficient constants are available in (23) and (27) to guarantee continuity of $f$ and $f'$ at $\xi = 0$, and thus solutions $\eta \in C^2$.

Now, the following information [2], [3], [4] concerning the properties of the solutions to Hill's equation will be useful at this time: For fixed values of $P$, $v$ in (25) there exist two monotonically increasing infinite sequences of real numbers of the form

$$\lambda_0, \lambda_1, \lambda_2, \lambda_3, \cdots \tag{28}$$

and

$$\lambda_1', \lambda_2', \lambda_3', \lambda_4', \cdots \tag{29}$$

Here $\lambda_n$ and $\lambda_n'$ satisfy the inequalities

$$\lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \cdots. \tag{30}$$

Let us now form the intervals

$$(-\infty, \lambda_0), (\lambda_0, \lambda_1'), (\lambda_1', \lambda_2'), (\lambda_2', \lambda_1), (\lambda_1, \lambda_2); (\lambda_2, \lambda_3'), \cdots. \tag{31}$$

The intervals (called instability intervals) of the type $(-\infty, \lambda_0)$, $(\lambda_{2n-1}, \lambda_{2n})$ are such that $\nu_1 \neq \nu_2$, Re $\nu_1 > 0$, Re $\nu_2 < 0$. On the other hand, in the intervals (called stability intervals) of the type $(\lambda_{2n}, \lambda_{2n+1}')$ and $(\lambda_{2n}', \lambda_{2n-1})$ we have $\nu_1 \neq \nu_2$, Re $\nu_i = 0$, $i = 1, 2$. Repeated values $\nu_1 = \nu_2$ occur at the end points: $\lambda_0$, $\lambda_1'$, $\lambda_2'$, $\lambda_1$, $\lambda_2$, $\cdots$. All intervals in (31) are finite with the exception of $(-\infty, \lambda_0)$.

In view of the above discussion it is evidently possible, for any values of $P$ and $v$, to choose $\sigma$ such that $\delta$ lies in an instability interval, i.e., in a region where one solution grows exponentially as $\xi \to -\infty$. In the $x$, $t$ space the separated solution $\eta(x, t)$ corresponding to this choice of $\sigma$ can be written

$$\eta(x, t) = u(x - vt) \exp [i\sigma(vx - t)/(v^2 - 1)] \tag{32}$$

where $u(x - vt)$ is defined by (27). Although (33) has unbounded initial values for this case, an inspection of the geometry of the problem indicates the Riemann function of the variational equation must grow exponentially in $t$ and thus any solution $\eta \in \mathfrak{F}$ will similarly have exponential growth in $t$. To see this, observe Fig. 3. According to (27) and (32) $|\eta|$ grows exponentially on the $x$ axis as $x \to -\infty$. However $|\eta|$ is constant along lines of $\xi = x - vt = $ constant. Considering $\eta(0, t)$ we see that the line $|\eta| =$ constant intersects the $x$-axis to the left of the domain of dependence of $\eta(0, t)$. Thus, $\eta(0, t)$ grows faster (exponentially) than the largest initial value in its domain of dependence and one must conclude that the Riemann function of the variational equation grows exponentially in $t$.

The above analysis for $v > 1$ follows closely a similar analysis by Ungar [5] who investigated the instability of wave trains associated with a certain one-dimensional wave equation related to Eq. (1). The foregoing arguments concerning the Riemann function can be found in a precise algebraic form in Ungar's paper.

Finally, a few remarks concerning the above approach is in order at this point. We have stated that a value of $\sigma$ can be selected such that $\lambda$ lies in an instability interval. This presupposes that the instability intervals do not vanish. In general, while the stability intervals can never vanish, the finite instability intervals can, in certain special cases, vanish. Recall, however, that we have assumed that either (1) $g(y)$ is a smooth function of $y$ with $g'(y)$ strictly decreasing with increasing $|y|$ or (2) $g(y)$ is not infinitely differentiable at $y = 0$. In the first case Ungar [5] has shown that $\sigma = 0$ is the left endpoint of an interval of instability of (25); in the second case a theorem by Hochstadt [4] indicates an infinite number of finite instability intervals exists, thus ensuring the possibility of selecting $\sigma$ such that $\lambda$ lies in an instability interval.

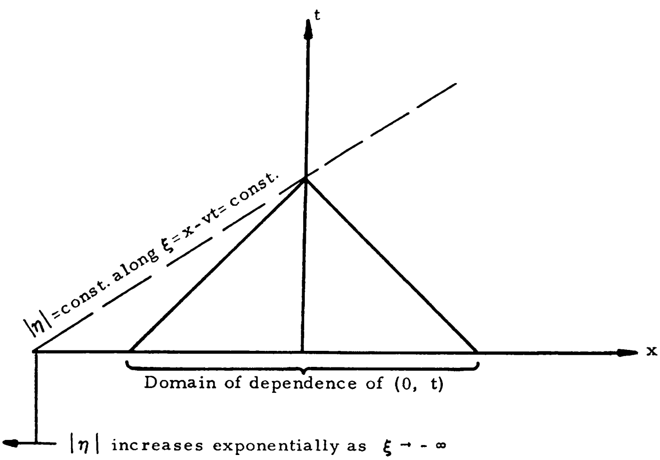


Fig. 3. Geometry in $x$, $t$ plane for $v > 1$.

*The Linear Case* $v > 1$. The above proof of instability for $v > 1$ fails if $g(y)$ is a linear function of $y$ (all the finite instability intervals vanish if $q \equiv 0$). In fact, $y_s(\xi)$ is *stable* for $v > 1$ when $g(y) = ky$, $k =$ positive constant. This can be seen as follows: in the $x$, $t$ plane the variational equation (14) has, for the linear $g(y)$ case, the form

$$\eta_{xx} - \eta_{tt} - k\eta = 0, \qquad k > 0. \tag{33}$$

Consider the energy functional

$$E^*(t) = \int_{-\infty}^{\infty} (\eta_x^2 + \eta_t^2 + k\eta^2)\, dx. \tag{34}$$

Equation (33) implies $dE^*(t)/dt = 0$ and thus $E^*(t) = E^*(0) =$ constant. As before, for $\eta \in \mathfrak{F}$, we can write, using the Cauchy–Schwartz inequality,

$$\eta^2 = \int_{-\infty}^{x} \frac{\partial}{\partial x} (\eta^2)\, dx \le 2\left[\int_{-\infty}^{\infty} \eta^2\, dx\right]^{1/2}\left[\int_{-\infty}^{\infty} \eta_x^2\, dx\right]^{1/2} \le \frac{(2)E^*(0)}{k} \tag{35}$$

from which we obtain

$$\sup{}_x |\eta| \le \left[\frac{(2)E^*(0)}{k}\right]^{1/2}.$$

Therefore, for $v > 1$ and $g(y) = ky$, $k > 0$, $y_s(\xi)$ is stable for all $P < \infty$ in the following sense: for any $\epsilon > 0$ we have $\sup_x |\eta| < \epsilon$ for all $t \ge 0$ provided $[(2)E^*(0)/k]^{1/2} < \epsilon$.

An interesting observation can be made at this point. Let $g(y)$ be written as

$$g(y) = ky + \mu\bar{g}(y), \qquad \mu > 0 \tag{36}$$

where $\bar{g}(y)$ represents the nonlinear portion of $g(y)$. Thus, for all values of $\mu > 0$, in particular as $\mu \to 0$, $y_s(\xi)$ is unstable for $v > 1$. However, if we set $\mu \equiv 0$, $y_s(\xi)$ is stable for $v > 1$ and all values of $P < \infty$. As far as stability is concerned, therefore, one observes a discontinuous behavior with respect to the system non-linearity parameter $\mu$ at $\mu = 0$.

**5. Conclusions.** Bounded, continuous steady-state (depending only on $x - vt$) solutions of the equation

$$y_{xx} - y_{tt} - g(y) + P\,\delta(x - vt) = 0$$

were considered. The function $g(y)$ was assumed to satisfy $g(0) = 0$ and $g'(y) \ge \delta > 0$, and except for the linear case $g(y) = ky$, $k > 0$, either (a) $g(y)$ is a smooth function of $y$ with $g'(y)$ strictly decreasing as $|y|$ increases or (b) $g(y) \in C^n$ where $n$ is finite and $n > 1$, but $g(y)$ is not infinitely differentiable at $y = 0$. The following items evolved from the analysis:

1. If $v < 1$ the steady-state solution, $y_s(x - vt)$, is uniquely dependent upon $P$, is a symmetric, positive function of $x - vt$ possessing a maximum at $x - vt = 0$, and is strictly decreasing to zero on either side of $x - vt = 0$ as $x - vt \to \pm\infty$. For $v > 1$ $y_s$ is zero for $x - vt \ge 0$ and is periodic function of $x - vt$ for $x - vt < 0$; again, $y_s$ depends uniquely upon $P$.

2. If $g(y)$ is a nonlinear function of $y$, then we have:
   (a) $y_s$ is locally stable for all $P < \infty$ when $v < 1$,
   (b) $y_s$ is locally unstable for all $P < \infty$ when $v > 1$ and thus $y_s$ cannot be expected as the limit of a transient motion when $v > 1$.

3. If $g(y)$ is a linear function of $y$, i.e., $g(y) = ky$, $k > 0$, then $y_s$ is locally stable for all $P < \infty$ in both regions $v < 1$ and $v > 1$.

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