—BOOK REVIEW SECTION—

Methods of moments in applied mathematics. By Yu. V. Vorobyev. Translated from the Russian by Bernard Seckler. Gordon and Breach Science Publishers, New York, London, Paris, 1965. x + 168 pp. $12.50.

In abstract form, the underlying principle of the Ritz-Galerkin type methods can be phrased as follows: Let $A: H \rightarrow H$ be an operator on the Hilbertspace H and $\{z_k\}_0^\infty$ a sequence of linearly independent elements which is linearly dense in H. Set $H_n = V_{k=0}^{n-1} z_k$ and define for $x = \sum_{k=0}^{\infty} a_k z_k$ the projection $P_n x = \sum_{k=0}^{n-1} a_k z_k$ onto H_n. Then, under suitable conditions, the sequence of operators $A_n = P_n A P_n$ represents increasingly better approximations of A. For the successful application of this concept the appropriate choice of the $\{z_k\}$ is of course a fundamental problem. The so called method of moments, discussed in this book, is a method of the above type where for each n the vectors $z_0, \ldots z_{n-1}$ spanning H_n are chosen as $z_k = A^k z_0$ $(k = 0, \ldots n-1)$ and where instead of the projection P_n the orthogonal projection E_n from H onto H_n is used to define A_n by $A_n = E_n A E_n$. For practical purposes, A_n is needed only on H_n and it can be shown that the reduction of A_n to H_n is uniquely determined by the relations $A^k{}_n z_2 = A^k z_0$, $k = 0, 1, \cdots, n-1$, $E_n z_n = A_n^n z_0$ provided $z_0, \cdots, z_n$ are linearly independent. It can also be shown that for bounded A the sequence of operators A_n converges strongly to A on the subspace $H_z = \overline{V_{k=0}^{\infty} A^k z_0}$ of H, and that for completely continuous A there is even convergence in norm. This represents in part a special case of known convergence results for the general Galerkin method.

For numerical purposes it is particularly important that — for example in the case of completely continuous A — the solutions x of $x = A\ x + b$ converge superlinearly to the solution x^* of $x = Ax + b$, and that a similar result holds for the corresponding eigenvalue problems. This fact makes the method very attractive in all those applications where the formulations of the approximating problems do not involve too many practical difficulties.

The method is of course closely related to the classical moment problem; for the case of symmetric matrices it represents a conjugate direction method, namely the Lanczos method of minimal iterations; when applied to eigenvalue problems for such matrices it differs from Krylov's method only in the computational scheme.

The book presents a survey of the theoretical aspects of the method, as investigated primarily by Russian mathematicians, as well as a discussion of its applications to a surprising variety of different problems. In an introductory Chapter I, concepts from Hilbert space theory are summarized and then the method itself is introduced for the case of bounded A. Chapters II and III discuss the theory and application for the cases of completely continuous and self-adjoint A. In Chapter IV the method is applied to the problem of accelerating the convergence of linear iterative processes, and as an example the Gauss-Seidel iteration for solving the finite difference analogues of an elliptic equation is considered. Chapter V presents applications to time dependent problems of the form $dx/dt + Bx = 0$ where $B: H \rightarrow H$ is an unbounded, positive definite, symmetric operator on a dense domain in H. This leads to applications for oscillatory systems. A brief Chapter VI goes into the possibility of extending the method to unbounded operators A, and, returning to the bounded case, the final Chapter VII gives applications to the solution of integral equations as well as to Sturm Liouville- and elliptic-boundary value problems.

The presentation is somewhat formalistic and not overly lucid, but should offer no difficulties to a reader with a knowledge of the essentials of the theory of operators in Hilbert space. The translation reads well. There are numerous detailed examples, frequently including numerical results. Altogether, this book certainly provides a heretofore lacking survey of an interesting method which does not appear to have received too much attention in this country, yet which clearly offers very useful numerical possibilities.

W. C. Rheinboldt (College Park, Md.)

Introduction to algebra. By Sam Perlis. Blaisdell Publishing Co., A Division of Ginn and Co., Waltham, Mass., Toronto, London, 1966. xx + 440 pp. $9.50.

In his preface the author states seven aims which in his opinion should be reached by teaching an algebra course of the kind he proposes "1. The student must be made aware of algebra not as a mass of manipulations but as an activity carried on in specified algebraic systems. 2. He must be made to feel at home with the concepts and elementary facts of five of these systems: groups, rings, integral domains, fields, and vector spaces. 3. He should be able to give specific examples of these systems. 4. The course should be used to wipe out pockets of ignorance concerning induction, roots of unity, multiple roots of polynomials, and irreducibility. 5. The student must master the basic facts of linear algebra, matrices, linear transformations. 6. Connections with elementary mathematics must be apparent to the student at almost every point. 7. Although some minimal list of subject matter could be compiled it seems more important to set upper bounds against an excess of difficult abstractions and deep theorems." Probably everybody will agree with these aims with regard to all students who plan to attend some advanced mathematics courses, pure mathematicians as well as students of physical sciences and engineering. Whether the author has reached his aims, will to a certain extent depend on the content, apart from the manner of presentation.

Chap. I. Fundamental concepts (p. 1-21), includes remarks on mathematical terminology and manner of expression. Chap. II (p. 22-68): Fields F of complex numbers and vector spaces of n-tuples over F; systems of linear equations and formal matrix operations. Chap. III (p. 69-105) has the simplest ideas of group theory. Chap. IV (106-147): Rings, with attention to quaternions, polynomials over a ring; indeterminates; ideals and homomorphism. Chap. V (148-169) Integral domains. Chap. VI. Fields. (p. 170-188). Chap. VII. Divisibility (p. 189-209) continues preceding discussion of integral domains, g.c.d., unique factorization. Chap. VIII (p. 210-241) is called Classical Algebra, sketches the definition of real numbers as equivalence classes of Cauchy sequences of rationals, complex numbers, roots of polynomials over a field, irreducibility (Eisenstein); the so-called fundamental theorem of algebra is stated together with the theorem on the existence of a closed extension for every field, both not proved. Chap. IX (p. 242-274) gives a more sophisticated approach to vector spaces, making use of matrices after introduction of the basis. Chap. X (275-313) Extension fields, essentially algebraic extensions, algebraic numbers, constructible numbers, finite fields, algebraic integers, and an example of an integral domain without unique factorization. Chap. XI. Determinants (p. 314-338). Chap. XII (p. 339-373) Linear transformations (which some readers would have liked to see earlier in the course) and similarity of matrices. Chap. XIII (p. 374-389) Forms and matrices. Chap. XIV (p. 390-430) Length and orthogonality. It might be added, that Chapters II, IX, XI-XIV should present a reasonable introduction to linear algebra for a student who has learned his elementary algebra.

The manner of presentation is essentially classical, thus desirable for students who do not wish to specialize in algebra. They will find the book easy to read, careful explanations wherever a difficulty arises, new concepts illustrated by simple, but not trivial examples, a lively style using picturesque language (e.g. on p. 258: "The availability of the dimension concept in vector spaces makes possible a small torrent of vital applications..."), plenty of exercise examples, none too hard, index of common symbols, review sections at the end of each chapter; all this should help to make an algebra course on these lines more popular among students whose primary interest lies in the applications of mathematical sciences. The algebraically gifted student will prefer a shorter text since many of the long-winded explanations will not be necessary for him.

H. Schwerdtfeger (Montreal, Que.)

Multidimensional singular integrals and integral equations. By S. G. Mikhlin. Translated from the Russian by W. J. A. Whyte, and translation edited by I. N. Sneddon. Pergamon Press, Oxford, London, Edinburgh, New York, Paris, Frankfurt, 1965. xii + 259 pp. $15.00.

In many problems of mathematical physics, especially those leading to partial differential equations of elliptic type, it is possible to convert the problem to an integral equation, e.g. with the help of a fundamental solution or a Green's function. The integral equation so obtained is often singular; in other words, the linear operator involved in it is not completely continuous, and the classical Fredholm solution is inapplicable.

A typical singular integral operator in m-dimensional Euclidean space E_m is one defined by the equation

$$v(x) = a_0(x)u(x) + \int_{E_m} \frac{f(x, \theta)}{|y - x|^m} u(y)\, dy,$$

where x and y denote points, $|y - x| = r$ is the distance between them, and $\theta = (y - x)/r$; the function f is required to satisfy

$$\int_S f(x, \theta)\, dS = 0,$$

where θ runs over the unit sphere S in E_m.

The author discusses the properties of these operators in considerable detail, including conditions for them to be bounded in some L_p space, their effects on the smoothness properties and on the asymptotic behavior of functions, their composition and differentiation, etc. Integral equations involving them are handled in a number of ways, especially by the application of a further operator, which may either give an immediate solution or *regularize* the equation, i.e. reduce it to a form for which elementary methods are available. Some results from general Banach space theory are found useful in this connexion.

A key role in the author's treatment is played by the *symbol* of a singular operator. This is a function with the property that to the composition of two operators corresponds the product of their symbols; in certain special circumstances, it can be expressed as a Fourier transform. The author discusses relations between the properties of an operator and those of its symbol, and the use of the symbol in finding regularizing operators.

Some sample applications of the theory to problems of mathematical physics are given in the last chapter. There is a historical introduction and a fairly full bibliography; some references to important papers appearing since the publication of the Russian edition have been inserted. The translation is intelligible, but betrays lack of acquaintance with mathematical terminology, and is inelegant and sometimes overliteral; on p. 63 'Excluding' is used instead of 'Eliminating,' and on p. 173 'homomorphic' for 'homeomorphic,' and the translator has not always guessed rightly whether to use the definite or indefinite article.

The book gives a useful picture of an important and rapidly developing field of mathematics, to which the author himself has made very substantial contributions.

F. Smithies (Cambridge, England)

Integral, measure and derivative: a unified approach. By G. E. Shilov and B. L. Gurevich. Revised English edition translated and edited by Richard A. Silverman. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966. xiv + 233 pp. $8.50.

This book is devoted to the development of the Lebesgue integral by approaching it from the Daniell integral angle. The latter is based on a fundamental class H of elementary bounded functions on an abstract set X with the class H a linear lattice and an elementary integral I which is a nonnegative (or positive) linear functional on H subject to the continuity condition that if $h_n(x)$ is a monotonic nonincreasing sequence in H converging to the zero function, then $\lim_n Ih_n = 0$. The Lebesgue type of integral then emerges by extension of the class H first by monotonic increasing sequences of functions of H and the extension of I to the larger class, and then indulging in a linear extension of the class and the operator. Here known properties of the Lebesque integral serve as a guide. The exposition of these ideas, the connection between the general integral and the Lebesgue integral and the properties of the Lebesgue integral are carried through in a lucid way. The book treats first the Riemann integral on an n-dimensional parallelopiped (called blocks) and indicates how the related upper and lower Darboux integrals can be obtained from a kind of integral on step functions, leading in short order to the basic necessary and sufficient condition for the existence of the Riemann integral in terms of the set of discontinuities of the function. There follows the definition of the Daniell integral, its properties and relation to the Lebesgue integral as a special case. A definition of a type of Riemann-Stieltjes integral based on left open blocks and an additive function on these yields by extension a Lebesgue-Stieltjes integral. Measure of a set is defined in terms of the integral of the characteristic function and measurable functions are pointwise limits of elementary functions almost everywhere. There is an abstract measure theory on semi-rings, rings and σ-rings of subsets of the abstract space. Three definitions of the derivative of a completely additive set function on a σ-ring of sets with respect to a completely additive measure are given, and extensive use of the Radon-Nikodym theorem is made to obtain relations between integral and derivative, the three definitions yielding the same function up to a set of zero measure under suitable conditions.

The introduction suggests that a background in advanced calculus would be an adequate preparation for the understanding of the book, but probably a course in functions of a real variable including Lebesgue integrals of one variable would be more to the point. For one familiar with the Lebesgue integral, the procedures of the book with emphasis on the abstract and occasional relapse to the concrete yield an interesting insight into classical Lebesgue theory. While it is suggested in the introduction that physicists might be interested in the exposition, there are no indications how applied mathematics enters the picture, not even the space L^2 of Lebesgue integrable square functions so important in the development of functions in series of orthogonal functions receives special mention. There are exercises at the end of each chapter, which would be more stimulating to the student if most of them were not supplied with Hints. References in the text are scarce and largely to Russian mathematicians, while the bibliography at the end is limited to related textbooks in English.

T. H. HILDEBRANDT (Ann Arbor, Mich.)

Numerical solution of initial value problems. By F. Ceschino and J. Kuntzmann. Translated by D. Boyanovitch. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966. xvi + 318 pp. $10.50.

In chapter 1 the authors introduce notational conventions, terminology (canonical forms, resolved forms, equilibrated resolved forms, etc.), brief statements of existence theorems (assuming functions are differentiable), and brief comments on numerical procedures.

Chapters 2–5 cover approximately 100 pages and are devoted mainly to deriving various single-step formulas, from Euler's method, through Runge-Kutta methods, to variants on these methods, such as one due to Blaess, and the implicit Runge-Kutta methods. Some Runge-Kutta formulas of order 5 and 6 are given, as well as a proof that formulas of order 5 cannot be obtained with only 5 function evaluations.

In chapter 6 (Adams Method and Analogues) some special multistep formulas are derived, including explicit and implicit Adams formulas, and those due to Cowell, Nyström, and Milne.

Chapter 7 (Different Multistep Formulas) is devoted mainly to a discussion of stability, without attempting to prove any of the basic theorems due to Dahlquist, even though both authors have made interesting contributions to this area in at least one of their earlier papers. Brief mention is also made of special formulas, such as those "which appeal to higher order derivations," and also some "of the Obrechkoff type."

Chapter 8 (Application of the Runge-Kutta Principle to the Multistep Methods) considers very briefly the idea of combining Runge-Kutta and multistep ideas into composite formulas.

Chapter 9 (Theoretical Considerations) consists mainly of remarks about the characteristic roots of linear homogeneous systems with constant coefficients, the "propagation matrix" for the variational equations, and the use of something called a "coaxial" in investigating the errors associated with various methods.

A final chapter (Practical Considerations) is concerned with a variety of topics, such as different ways of estimating local truncation errors, choosing the step-size, and changing the step-size.

Numerical results for relatively simple problems are used frequently, in almost every chapter, to illustrate the methods being discussed.

There is a bibliography of nearly 600 items, including a few for 1962. (The text from which the translation has been made was copyrighted in 1963).

This book provides a large number of formulas for the numerical integration of ordinary differential equations. Unfortunately, despite the claim in the preface that the authors have tried to group the methods around a central idea, the result is a hodge-podge of formulas, facts, near-facts, and opinions. The treatment is sometimes incomplete, and often superficial.

The presentation is frequently rather vague or misleading. For example, when "methods of approximate solution" are first introduced on page 11, it is stated that "These methods do not give the general integral but only a well-determined integral. These methods can furnish, instead of the exact solution which we do not know or do not wish to write down, an approximate solution in the sense of numerical calculus. We thus understand that this solution is defined in a finite interval by a procedure actually executable and that we possess certain information on the error by which it is affected." Does this help the reader to understand the nature of numerical methods?

To illustrate some of the carelessness with which this book is written, consider the way in which the method of Euler-Cauchy is introduced, after the tangent method has been described, and subsequently improved. On page 40, it is stated that "Another manner of improving the tangent method consists of noting that, on a small arc, the slope of the chord is obviously the arithmetic mean of the slopes of the tangents at the end points."

The English is not good, and the fact that the book is a translation is frequently obvious. But the main difficulties with the presentation must have existed in the original version.

T. E. HULL (Toronto, Ont.)

Introduction to computing. By T. E. Hull. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1966. xi + 212 pp. $6.95.

The major part of this well-written book is concerned with FORTRAN IV. In Chapter 1, algorithms and flow charts are discussed. Chapter 2 describes a hypothetical stored-program computer. Some of the algorithms discussed in Chapter 1 are implemented on this machine and the correspondence between flow-chart and machine-language descriptions is stressed. Chapter 3 is concerned with basic programming techniques (in machine language). Chapter 4 deals with translators and compilers. Chapters 5 through 10 discuss FORTRAN IV (Fortran Programs—Constants and Variables—Expressions and Assignment Statements—Control Statements—Input and Output Statements—Subprograms). Chapter 11 is concerned with program planning and debugging. Chapters 12 and 13 deal with numerical methods and non-numerical applications. Chapter 14 is devoted to simulation techniques and Chapter 15 discusses algorithms, automata, and languages. The material is very clearly presented.

W. PRAGER (La Jolla, Cal.)

Relativistic kinematics. By Henri Arzeliès. Pergamon Press, Oxford, London, Edinburgh, New York, Toronto, Paris, Frankfurt, 1966. xi + 298 pp. $12.00.

The first edition of this work appeared in French in 1955. The present book is translated from an enlarged and thoroughly revised text, and is brought up to date to include the latest publications. The author claims to have reduced the mathematical formalism to the necessary and sufficient minimum for the physical theory: "The essential results can and must be derived by means of simple mathematics. The idea that the theory of relativity can be properly presented only in the language of four-vectors is too widespread and should be resisted. Such a screen of notation often prevents many readers from appreciating the essence of the theory." In this spirit he deals carefully and in detail with topics arising out of Einstein's formulation (1905) of the special theory of relativity. Each chapter is followed by historical and bibliographical notes, and here the good-natured wit of Professor Arzeliès embellishes his criticisms of confused thinking. He narrates (p. 190) that he corresponded with Professor Dingle about the famous clock paradox: "—the exchange was pleasant and instructive, but without result. Dingle displays what I believe the theologians call "an invincible blindness." Only prayer can help him; and whatever some people may care to think, relativity is not a religion, nor does it have priests."

Professor Arzeliès relegates the Minkowskian continuum to the last chapter. Had he put it first, and used it as a basis of discussion, then the many curious controversies of which he writes would have appeared in their true light as so many mares' nests. Since the history of human folly is interesting, it is good to have this aspect of it treated with such skill and sympathy, and from this point of view the book deserves high praise. But I hesitate to recommend it as a textbook for the physicist of 1966 who wants to learn about the special theory of relativity. Must he too have his bones broken on the quasi-Newtonian wheel on which his grandfather suffered? Far better recognise the fact that time marches on and the physicist of today must straighten out his thoughts by resorting to slightly more sophisticated mathematics. The special theory of relativity is nothing but the Lorentz group with physical frills on it, or, if you are geometrically disposed, for "Lorentz group" read "Minkowskian space-time". Master that, and you will save yourself from controversies which can never be resolved because the antagonists start from different ill-defined premises.

J. L. SYNGE (Toronto, Ont.)