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CONTINUITY CONDITIONS AT WAVE FRONTS IN COUPLED THERMOPLASTICITY

BY

A. D. FINE

Pratt & Whitney Aircraft, East Hartford, Connecticut

Abstract. The mass, momentum, and energy conservation requirements at wave fronts (or surfaces of discontinuity) are reduced to a form suitable for use in the coupled thermoplastic analysis. Constitutive equations are introduced in these relations, and, for one-dimensional situations, it is shown that an isothermal (or adiabatic) wave front separating regions of elastic and plastic behavior will propagate with the same speed as an isothermal (or adiabatic) surface of discontinuity lying completely inside a domain of plastic deformation. A new thermoplastic coupling parameter is obtained from an example involving spherical symmetry.

1. Introduction. In recent years, many studies have been carried out in an attempt to describe the behavior caused by the thermal changes in a solid body. Starting with the linear *un*coupled thermo*elastic* theory [1], [2], these analyses have involved, in increasing complexity, linear coupled thermo*elasticity* [3], *un*coupled thermo*plasticity* [4], and coupled thermo*plasticity* [5]. In this paper which deals with the coupled thermoplastic theory, we consider the conditions (in particular, the energy conservation requirement) which must be imposed at wave fronts (surfaces of discontinuity), that is, at surfaces across which the displacement is continuous, but where the temperature as well as the first derivatives of the displacement may be discontinuous. We show that the jump conditions appropriate at such fronts in the aforementioned theories are actually included in our more general formulation.

In Sec. 2, we present the basic restrictions (sometimes called jump conditions or discontinuity relations) that must be imposed at wave fronts. After reducing the restrictions to a form convenient for our analysis, we introduce in Sec. 3 the constitutive equations which are then used to simplify further the discontinuity relations. An example involving spherical symmetry is given in Sec. 4. It is shown that, in the coupled thermoplastic theory, the conditions applicable at an EP-surface, i.e., at a surface of discontinuity separating regions of elastic and plastic behavior, are identical to those conditions that are imposed at plastic wave fronts, i.e., at surfaces of discontinuity lying entirely within the domain of plastic deformation. In Sec. 5, we discuss these results and we show that a new thermal coupling parameter has been obtained for the thermoplastic analysis.

2. Basic equations. We consider an elastic-perfectly plastic body which is homogeneous, isotropic, and at a uniform temperature T_0 in its undeformed state. The com-

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ponents of the displacement vector at time t are denoted by $u_i(x_j, t)$, where x_i are the coordinates of a typical point of the body in a fixed rectangular Cartesian coordinate system and the Latin indices range over the values 1, 2, and 3. The temperature is denoted by $T(x_i, t)$.

The stress and strain components referred to the Cartesian coordinate system are written, respectively, as $\sigma_{ij}(x_m, t)$, and $\epsilon_{ij}(x_m, t)$, where

$$\epsilon_{ij}(x_m, t) = \frac{1}{2}\left\{\frac{\partial u_i(x_m, t)}{\partial x_j} + \frac{\partial u_j(x_m, t)}{\partial x_i}\right\}, \tag{1}$$

while the components of the stress and strain deviators are denoted by $s_{ij}(x_m, t)$ and $e_{ij}(x_m, t)$ respectively and are of the form

$$\begin{aligned} s_{ij}(x_m, t) &= \sigma_{ij}(x_m, t) - \tfrac{1}{3}\sigma_{kk}(x_m, t)\,\delta_{ij}, \\ e_{ij}(x_m, t) &= \epsilon_{ij}(x_m, t) - \tfrac{1}{3}\epsilon_{kk}(x_m, t)\,\delta_{ij}. \end{aligned} \tag{2}$$

Here δ_{ij} is the Kronecker delta and a repeated index implies summation over the values 1, 2, and 3.

As is well known in continuum mechanics, there may exist, in a body, moving surfaces across which the displacement is continuous, but where the temperature as well as the first derivatives of the displacement are discontinuous [6]. We call such a surface a wave front or surface of discontinuity and denote it by S, Fig. 1(a). The unit normal vector to S at P_0 in the direction of propagation of the surface is written as $\mathbf{n}$, with components n_i, while two unit tangents are denoted by $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, with components μ_i and ν_i, where $\mathbf{n}$, $\boldsymbol{\mu}$, and $\boldsymbol{\nu}$ form a right-handed orthogonal triad. Jumps in value of quantities across S at P_0 will be indicated by the notation

$$[Q] = Q^2 - Q^1,$$

where a superscript 1 or 2 denotes the limiting value of a quantity as the surface is approached from region 1 or 2 respectively.

We now impose at S a set of conditions, sometimes called jump conditions or discontinuity relations, which provide that mass, momentum, and energy be conserved across S. We first consider the energy conservation requirement which can be written as [7]

$$[\sigma_{ij}\dot{u}_i n_j - q_i n_i - \rho(e + \tfrac{1}{2}\dot{u}_j\dot{u}_j)(\dot{u}_i n_i - V)] = 0, \tag{3}$$

where ρ is the density, e the internal energy per unit mass, q_i the heat flux vector, and V the normal speed of propagation of S in the $\mathbf{n}$ direction; the notation used is

$$Q_{,i} = \frac{\partial Q(x_m, t)}{\partial x_i}, \qquad \dot{Q} = \frac{\partial Q(x_m, t)}{\partial t}.$$

We introduce here, as has been done before [1], [2], the free energy function F by the relation

$$e = F + T\eta, \tag{4}$$

where η is the entropy density per unit volume. Assuming that F depends explicitly on temperature, as well as on certain other quantities not involving time rates, we can represent η in terms of F by, [1], [5],

$$\eta = -\partial F/\partial T. \tag{5}$$

If we put (5) into (4) and use the resulting expression in (3), we obtain

$$[\sigma_{ij}\dot{u}_i n_j - \rho(F + \tfrac{1}{2}\dot{u}_j\dot{u}_j)(\dot{u}_i n_i - V) + \rho T(\partial F/\partial T)\dot{u}_i n_i] = [q_i n_i + \rho VT(\partial F/\partial T)]. \tag{6}$$

We are interested in the elastic-perfectly plastic continuum whose behavior is described by the linear coupled thermoelastic theory [1] in the region of elastic behavior and by linear coupled thermal equations in the region of plastic behavior. We therefore neglect all terms which are not linear in the dependent variables. In particular, we replace ρVT by ρVT_0 on the right side of (6) and we consider ρ as constant. Since all the factors appearing on the left side of (6) are nonlinear in the dependent variables, we can reduce Eq. (6) to the form

$$[q_i n_i + \rho VT_0(\partial F/\partial T)] = 0. \tag{7}$$

Equation (7) thus represents at a wave front the linearized energy conservation condition for any material which admits a free energy function and for which the thermodynamic relation (5) is valid. The other thermodynamic restriction at a surface discontinuity is the entropy production jump condition which can be written in the general case as (see [7, Equation (25)])

$$\left[\frac{q_i n_i}{T} + \rho\eta(\dot{u}_i n_i - V)\right] \geq 0.$$

If we then use Eq. (5), this inequality may be reduced to the more convenient form

$$\left[\frac{q_i n_i}{T_0} + \rho V\frac{\partial F}{\partial T}\right] + \left[q_i n_i\left(\frac{1}{T} - \frac{1}{T_0}\right) - \rho\dot{u}_i n_i\frac{\partial F}{\partial T}\right] \geq 0.$$

From Eq. (7), however, we see that the first bracketed expression must vanish. Consequently, the entropy production jump condition at a wave front involves only quantities of the second order and higher in the dependent variables, and yields therefore no additional restrictions in the linearized theory.

The remaining requirements at S for the conservation of momentum and mass and for the continuity of displacement can be described, in linearized form [8], by

$$[\sigma_{ij}n_j + \rho V\dot{u}_i] = 0, \tag{8}$$

$$[\dot{u}_i n_j + Vu_{i,j}] = 0. \tag{9}$$

3. Constitutive relations. We now investigate the functional form of the free energy F. For the coupled thermo*elastic* theory, F is assumed to depend only upon the strain tensor ϵ_{ij} and the temperature T, and consequently to satisfy [1],

$$\rho F = \frac{\lambda}{2}\epsilon_{jj}\epsilon_{ii} + \mu\epsilon_{ij}\epsilon_{ij} - (3\lambda + 2\mu)\alpha T\epsilon_{jj} - \frac{\rho c}{2T_0}T^2. \tag{10}$$

Here λ and μ are the isothermal Lamé constants, α is the linear coefficient of thermal expansion, and c is the specific heat in the initial undeformed state. The stress tensor σ_{ij} and the free energy are related by [1],

$$\sigma_{ij} = \rho\,\partial F/\partial\epsilon_{ij}\,, \tag{11}$$

so that from (10) we obtain the stress-strain equation

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\,\delta_{ij} - (3\lambda + 2\mu)\alpha T\,\delta_{ij}\,. \tag{12}$$

For coupled thermal behavior in a domain of *plastic* deformation, F is assumed to depend only upon the dilation ϵ_{jj}, the stress deviator tensor s_{ij}, and the temperature T [5]. We assume in addition for simplicity that no unloading can occur in the region of plastic deformation and, as a result, we require that F reduce to the form presented in (10) if there is only elastic behavior. It is then possible to show in a fairly straightforward manner that the form of F applicable for this coupled thermo*plastic* analysis can be written as

$$\rho F = \frac{(3\lambda + 2\mu)}{6}(\epsilon_{ii}\epsilon_{jj} - 6\alpha T\epsilon_{ii}) + \frac{s_{ij}s_{ij}}{4\mu} - \frac{\rho c}{2T_0}T^2. \tag{13}$$

We note that the expression for $\partial F/\partial T$ derivable from (10) is identical to that derivable from (13) (if we recall (2)) and is given by

$$\frac{\partial F}{\partial T} = -\frac{(3\lambda + 2\mu)}{\rho}\alpha\epsilon_{jj} - \frac{c}{T_0}T. \tag{14}$$

If we now use the well known Fourier heat conduction law,

$$q_i = -kT_{,i}\,, \tag{15}$$

together with relation (14) in Eq. (7), we obtain the following desired form of the energy conservation condition across a surface of discontinuity:

$$[kT_{,i}n_i + \alpha T_0V(3\lambda + 2\mu)\epsilon_{jj} + \rho cVT] = 0, \tag{16}$$

where k is the thermal conductivity in the initial undeformed state.

It has been shown [5], in addition, that the relations,

$$\sigma_{ii} = 3\rho\,\partial F/\partial\epsilon_{ii}\,, \tag{17}$$

$$s_{ij} = 2\mu\rho\,\partial F/\partial s_{ij}\,, \tag{18}$$

are valid in coupled thermoplasticity, so that if we substitute (13) into (17), we obtain the equation

$$\sigma_{ii} = (3\lambda + 2\mu)(\epsilon_{jj} - 3\alpha T). \tag{19}$$

Substitution of (13) into (18) however results only in an identity. We also select here a yield condition that will be consistent with previous statements. We choose the Von–Mises criterion [9] which specifies that the behavior is elastic at a point of a body unless

$$s_{ij}s_{ij} = 2b^2 \quad\text{and}\quad s_{ij}\dot{s}_{ij} = 0, \tag{20}$$

where b is a constant and represents the known yield stress in shear.

4. Spherical wave fronts. As an example, we will consider the elastic-perfectly plastic continuum for which the deformation has complete spherical symmetry. We measure all quantities with respect to a fixed Cartesian coordinate system whose origin coincides with the center of symmetry. All quantities therefore are functions only of the time t and the distance r from the origin. We may therefore introduce a radial displacement $u(r, t)$ by means of the relation

$$u_i(x_j, t) = u(r, t)n_i, \tag{21}$$

where

$$n_i = x_i/r, \tag{22}$$

$r^2 = x_i x_i$, and n_i represent the components of a unit normal vector $\mathbf{n}$ to a surface of discontinuity, Fig. 1(b).

We employ the following notation in this section:

$Q(r, t)$ = a dependent variable associated with the radial ($\mathbf{n}$) direction;
$\bar{Q}(r, t)$ = a corresponding dependent variable associated with the tangential ($\boldsymbol{\mu}$ or $\boldsymbol{\nu}$) direction.

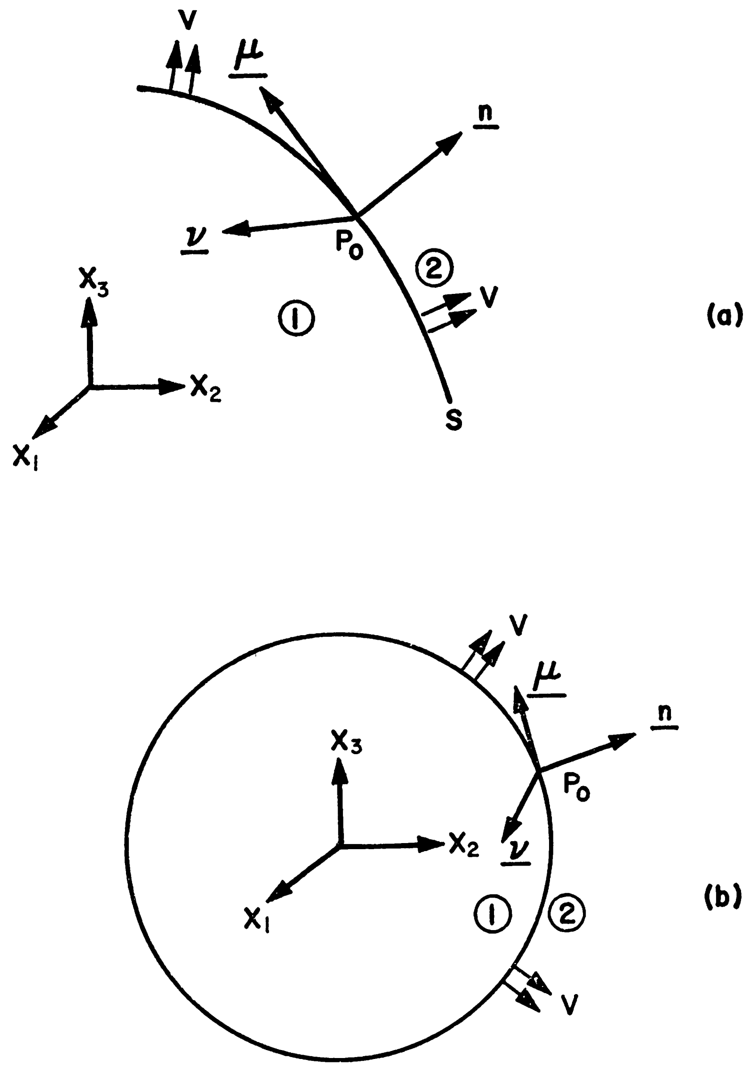

Fig. 1. (a) General Wave Front
(b) Spherical Wave Front

The notation for derivatives is similar to that of the previous sections. By using expression (21) together with (22) in Eqs. (1) and (2), we have the relations

$$\epsilon = u_r , \qquad \bar{\epsilon} = u/r, \tag{23}$$

$$s = \tfrac{2}{3}(\sigma - \bar{\sigma}), \qquad \bar{s} = \tfrac{1}{3}(\bar{\sigma} - \sigma). \tag{24}$$

For the region of elastic behavior, Eq. (21) can be substituted into the stress-strain relation (12) to give, with the aid of (22),

$$\sigma = \rho C_T^2 u_r + 2\rho(C_T^2 - 2C_S^2)\frac{u}{r}, \tag{25}$$

$$\bar{\sigma} = \rho(C_T^2 - 2C_S^2)u_r + 2\rho(C_T^2 - C_S^2)\frac{u}{r}, \tag{26}$$

Here C_T is the isothermal dilatational wave speed and C_S is the shear wave speed, with C_T and C_S defined by

$$C_T^2 = \frac{\lambda + 2\mu}{\rho}, \qquad C_S^2 = \frac{\mu}{\rho}. \tag{27}$$

By observing that condition (20) must be valid on both sides of the EP-surface, we find, with the help of (24)–(26), that

$$u_r^{\text{YEL}} - \frac{u^{\text{YEL}}}{r} = \pm\frac{3\nu K_T}{2C_S^2}, \tag{28}$$

where

$$\begin{aligned} 3K_T^2 &= 3C_T^2 - 4C_S^2 , \\ \nu &= b/\rho K_T 3^{1/2}. \end{aligned} \tag{29}$$

The symbol Q^{YEL} (or Q^{YPL}) represents here the limiting value of Q as the EP-surface is approached from the elastic (or plastic) side.

It is possible, because of the symmetry for this case, also to obtain simple stress-strain relations in the region of plastic behavior. By using expression (21) in Eqs. (19) and (20) and recalling (22)–(24), we get the following stress-strain equations:

$$\sigma = \rho K_T^2\left(u_r + 2\frac{u}{r} - 3\alpha T\right) \pm 2\nu K_T\rho, \tag{30}$$

$$\bar{\sigma} = \rho K_T^2\left(u_r + 2\frac{u}{r} - 3\alpha T\right) \mp \nu K_T\rho, \tag{31}$$

If we now introduce Eqs. (21)–(23), together with displacement continuity explicitly, into Eq. (16), we reduce the energy conservation requirement to the form

$$[kT_r + 3\alpha T_0 K_T^2 u_r + \rho c V T] = 0, \tag{32}$$

where $T = T(r, t)$. This result (32) is valid, moreover, at all spherical wave fronts, not only at elastic wave fronts (i.e., surfaces of discontinuity lying entirely within the region of elastic behavior) and at plastic wave fronts, but also at EP-surfaces of discontinuity. Jump relation (9), modified by Eqs. (21) and (22), likewise takes a form valid at all spherical surfaces of discontinuity, namely

$$[\dot{u} + Vu_r] = 0. \tag{33}$$

The momentum conservation requirement (8) becomes, for the spherically symmetric case,

$$[\sigma + \rho V\dot{u}] = 0. \tag{34}$$

At the EP-wave front, σ is given by (25) on the elastic side and by (30) on the plastic side of the front, so that Eq. (34), divided by ρ, can be written

$$K_T^2\left(u_r^{\text{YPL}} + 2\,\frac{u^{\text{YPL}}}{r} - 3\alpha T^{\text{YPL}}\right) \pm 2\nu K_T + V\dot{u}^{\text{YPL}}$$
$$= C_T^2 u_r^{\text{YEL}} + 2(C_T^2 - 2C_S^2)\,\frac{u^{\text{YEL}}}{r} + V\dot{u}^{\text{YEL}}. \tag{35}$$

By using yield condition (28), continuity condition (33), as well as $u^{\text{YEL}} = u^{\text{YPL}}$ explicitly, in (35), we obtain, with the aid of (29), the equation

$$(K_T^2 - V^2)[u_r] = 3\alpha K_T^2[T]. \tag{36}$$

The validity of this result then can easily be extended to plastic wave fronts. A similar expression,

$$(C_T^2 - V^2)[u_r] = 3\alpha K_T^2[T], \tag{37}$$

valid for elastic surfaces of discontinuity, may likewise be deduced from our general formulation, although this result (37) has been derived before [6].

Since the energy discontinuity relation (32) is valid at all spherical wave fronts, we may combine it with expressions (36) and (37) to obtain the following two equations:

$$\rho c V\{V^2 - K_T^2(1 + \phi^P)\}[u_r] = 3\alpha k K_T^2[T_r], \tag{38}$$

$$\rho c V\{V^2 - C_T^2(1 + \phi^E)\}[u_r] = 3\alpha k K_T^2[T_r]. \tag{39}$$

The first is valid at plastic surfaces of discontinuity as well as at EP-wave fronts, while the second is valid at elastic wave surfaces. The constants ϕ^P and ϕ^E are given by

$$\phi^P = q\alpha^2 T_0 K_T^2/c, \tag{40}$$

$$\phi^E = 9\alpha^2 T_0 K_T^4/cC_T^2. \tag{41}$$

5. Discussion of the results. We consider now the consequences of our foregoing analysis. At *plastic* as well as at EP-surfaces of discontinuity, where Eqs. (36) and (38) are valid, an isothermal wave front, i.e., $[T] = 0$, will result when $V^2 = K_T^2$, while an adiabatic front, i.e., $[T_r] = 0$, arises when $V^2 = K_T^2(1 + \phi^P)$. The isothermal and adiabatic *elastic* surfaces of discontinuity are seen, from Eqs. (37) and (39), to occur in turn when $V^2 = C_T^2$ and $V^2 = C_T^2(1 + \phi^E)$. The constants ϕ^P and ϕ^E therefore may be considered as the thermal coupling parameters respectively for the coupled thermoplastic and coupled thermoelastic theories.

If we now compare these two quantities by taking their ratio, we have, from (40) and (41),

$$\phi^P/\phi^E = C_T^2/K_T^2\,. \tag{42}$$

By using (29) in (42) and then dividing the result by $C_T^2/3$, we reduce (42) to

$$\phi^P/\phi^E = 3/(3 - 4\gamma), \tag{43}$$

where $\gamma = C_S^2/C_T^2$. It can easily be shown that γ must lie in the range [9]

$$0 < \gamma < 1/2,$$

and consequently, from (43), that the ratio ϕ^P/Q^E obeys the restriction

$$1 < \phi^P/\phi^E < 3.$$

We conclude therefore that, if a coupling effect is to be included, for a problem, in regions of thermo*elastic* behavior, such an effect must also be included, for this problem, in regions of thermo*plastic* behavior.

From Eqs. (36)–(39), it is easy to see that isothermal or adiabatic wave fronts, i.e., $[T] = 0$ or $[T_{,r}] = 0$, will propagate at certain fixed speeds. As discussed in [6], however, neither of these two conditions necessarily hold true at a surface of discontinuity. On the other hand, it has been asserted [10] that the continuity of temperature appears, from purely mathematical considerations, in linear coupled thermoelasticity, to be a necessary condition for the uniqueness of solutions of mixed initial and boundary-value problems. Consequently, although the trend in present thinking seems to point towards a continuous temperature field, the value which V does take in the general case still remains an open question.

We wish also to reemphasize that the energy conservation requirement (7) can be used for the linear theory at surfaces of discontinuity in any material where the entropy η, the free energy F, and the temperature T obey the thermodynamic relation (5). It is important finally to note that, as was shown in our example and as may be shown in general one-dimensional situations, the mass, momentum, and energy conservation requirements reduce to exactly the same form at EP-wave fronts as at plastic wave fronts.

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