—NOTES—

A NOTE ON SINGULARITIES IN A COSSERAT CONTINUUM*

By YECHIEL WEITSMAN (*Brown University*)

Abstract. This paper is concerned with the singularities that are due to concentrated couples in an infinite linear, elastic, isotropic Cosserat continuum. The solution to the problem of a concentrated couple, acting within an infinite region, may be obtained as a limiting case of the solution to the problem of body moments acting within a finite portion of the infinite medium. Alternatively, the solution to the problem of the concentrated couple can be constructed from the solution to the case of a concentrated force acting within an infinite body, by combining two double-forces with moments to form a center of rotation.

In this paper it is shown that in a Cosserat continuum the two above mentioned singular solutions to the case of a concentrated couple, acting within an infinite body, are not the same. By means of a specific linear combination of these two singular solutions it is possible to reconstruct the classical center of rotation, which is accompanied by an additional micro-rotation field. It is shown that there exists a limiting case in which the macro-displacements are eliminated altogether, resulting in a singular field of micro-rotations alone.

1. The equations of a linear, elastic, isotropic Cosserat continuum. In a Cosserat continuum [1]**, deformations are characterized by two kinematical variables: the displacement u_i and the independent, rigid, anti-symmetric micro-rotation $\psi_{[ij]}$. The quantities $\psi_{[ij]}$ describe a rigid rotation of some material "superstructural" property (e.g. the Cosserat triad or, alternatively, a "micro-structure").

Following Mindlin's formulation [2] of the linear, elastic case we define

$$\begin{aligned} \epsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ \gamma_{[ij]} &= \tfrac{1}{2}(u_{j,i} - u_{i,j}) - \psi_{[ij]} , \\ \kappa_{i[jk]} &= \psi_{[jk],i} . \end{aligned} \tag{1}$$

Then, for an isotropic, centrosymmetric medium the constituitive relations are

$$\begin{aligned} \tau_{ij} &= \lambda\epsilon_{kk}\,\delta_{ij} + 2\mu\epsilon_{ij} , \\ \sigma_{[ij]} &= 2\beta\gamma_{[ij]} , \\ \mu_{i[jk]} &= \alpha_1(\kappa_{l[lj]}\,\delta_{ik} + \kappa_{l[kl]}\,\delta_{ij}) + 2\alpha_2\kappa_{i[jk]} + \alpha_3(\kappa_{k[ij]} + \kappa_{j[ki]}), \end{aligned} \tag{2}$$

where τ_{ij} is the classical "Cauchy" stress, $\sigma_{[ij]}$ is the anti-symmetric part of Mindlin's relative stress, and $\mu_{i[jk]}$ is the Cosserat's couple-stress. The quantities α_i and β are

*Received February 21, 1966; revised manuscript received May 23, 1966.

**Numbers in square brackets indicate the reference listed at the end of this paper.

some material constants. The dimension of α_i differs from the dimension of the Lamé constants λ, μ by the square of length.

The equations of equilibrium read:

$$(\tau_{ij} + \sigma_{[ij]})_{,i} + f_j = 0,$$
$$\mu_{i[jk],i} + \sigma_{[jk]} + \phi_{[jk]} = 0. \tag{3}$$

In (3) f_j and $\phi_{[jk]}$ denote body force and body couple, respectively.

The boundary conditions, at a boundary with outward normal n_i, are

$$t_j = n_i(\tau_{ij} + \sigma_{[ij]}),$$
$$T_{[jk]} = n_i \mu_{i[jk]} . \tag{4}$$

Substituting (1) in (2), and then in (3) we obtain the kinematical equations of motion

$$(\lambda + \mu - \beta)u_{j,ji} + (\mu + \beta)u_{i,jj} - 2\beta\psi_{[ji],j} + f_i = 0,$$
$$(\alpha_1 + \alpha_3)(\psi_{[ki],kj} + \psi_{[jk],ki}) + 2\alpha_2\psi_{[ij],kk} - 2\beta\psi_{[ij]} + \beta(u_{j,i} - u_{i,j}) + \phi_{[ij]} = 0. \tag{5}$$

Employing the "direct" notation, Eqs. (5) read

$$(\lambda + \mu - \beta)\nabla\nabla\cdot\mathbf{u} + (\mu + \beta)\nabla^2\mathbf{u} - 2\beta\nabla\cdot\boldsymbol{\psi}^A + \mathbf{f} = 0,$$
$$(\alpha_1 + \alpha_3)(\nabla\cdot\boldsymbol{\psi}^A\nabla + \nabla\boldsymbol{\psi}^A\cdot\nabla) + 2\alpha_2\nabla^2\boldsymbol{\psi}^A - 2\beta\boldsymbol{\psi}^A + \beta(\nabla\mathbf{u} - \mathbf{u}\nabla) - \tfrac{1}{2}\mathbf{I}\times\mathbf{c} = 0. \tag{6}$$

In (6) the quantities $\mathbf{u}$ and $\mathbf{f}$ are vectors with components u_i and f_i and $\boldsymbol{\psi}^A$, $\boldsymbol{\phi}^A$ and $\mathbf{I}$ are dyadics with components $\psi_{[ij]}$, $\phi_{[ij]}$ and δ_{ij}. The quantity $\boldsymbol{\phi}^A$ has been written in terms of a body couple vector $\mathbf{c}$,

$$\boldsymbol{\phi}^A = -\tfrac{1}{2}\mathbf{I}\times\mathbf{c}.$$

Mindlin has shown [2], that a complete solution of (6), or (5), can be expressed as

$$\mathbf{u} = \nabla\times\mathbf{K} + (1 - l_3^2\nabla^2)(\mathbf{B} - l_1^2\nabla\nabla\cdot\mathbf{B}) - \tfrac{1}{2}(k_1 - l_3^2\nabla^2)\nabla[\mathbf{r}\cdot(1 - l_1^2\nabla^2)\mathbf{B} + B_0],$$
$$\boldsymbol{\psi}^A = -\tfrac{1}{4}\mathbf{I}\times[\nabla^2\nabla(\mathbf{r}\cdot\mathbf{K} + K_0) + 2\nabla\times\mathbf{B}] \tag{7}$$

where $\mathbf{B}$, B_0, $\mathbf{K}$, and K_0 are stress-functions of the Boussinesq–Papkovitch type. These functions satisfy the following relations:

$$\mu(1 - l_1^2\nabla^2)\nabla^2\mathbf{B} = -\mathbf{f} - \frac{\mu + \beta}{2\beta}\nabla\times\mathbf{c},$$
$$\mu\nabla^2 B_0 = \mathbf{r}\cdot\left[\mathbf{f} + \frac{\mu + \beta}{2\beta}\nabla\times\mathbf{c}\right],$$
$$2\beta\nabla^2\mathbf{K} = \mathbf{c},$$
$$2\beta(1 - l_2^2\nabla^2)\nabla^2 K_0 = 4l_2^2\nabla\cdot\mathbf{c} - \mathbf{r}\cdot(1 - l_2^2\nabla^2)\mathbf{c} \tag{8}$$

where, in (7) and (8)

$$k_1 = \frac{\lambda + \mu}{\lambda + 2\mu},$$
$$l_1^2 = (2\alpha_2 - \alpha_1 - \alpha_3)\frac{\mu + \beta}{2\mu\beta},$$
$$l_2^2 = \frac{\alpha_2}{\beta}, \qquad l_3^2 = \frac{2\alpha_2 - \alpha_1 - \alpha_3}{2\beta},$$

2. The singular solutions for concentrated force and couple. Mindlin [2] has given the solution to the problems of a concentrated force and a concentrated couple acting within an infinite medium.

For a concentrated force **P**, acting at the origin of a Cartesian coordinate system x, y, z the stress-functions are

$$\begin{aligned} \mathbf{B} &= \frac{\mathbf{P}}{4\pi\mu} g_1 , \\ B_0 &= 0, \\ \mathbf{K} &= 0, \\ K_0 &= 0. \end{aligned} \tag{9}$$

For a concentrated moment **C**, acting at the origin the stress functions are

$$\begin{aligned} \mathbf{B} &= -\frac{\mu + \beta}{8\pi\mu\beta} \mathbf{C} \times \nabla g_1 , \\ B_0 &= 0, \\ \mathbf{K} &= -\frac{\mathbf{C}}{8\pi\beta r} , \\ K_0 &= -\frac{l_2^2}{4\pi\beta} \mathbf{C} \cdot \nabla g_2 . \end{aligned} \tag{10}$$

In (9) and (10) $g_i = (1 - e^{-r/l_i})/r$.

Let $\mathbf{u}^{(1)}$ and $\psi^{A(1)}$ denote the kinematical field due to a concentrated force $P\mathbf{e}_x$ acting at the origin, and $\mathbf{u}^{(2)}$, $\psi^{A(2)}$ be the kinematical field that is due to a force $P\mathbf{e}_y$ acting at the origin.

Employing the fields $\mathbf{u}^{(1)}$, $\psi^{A(1)}$ and $\mathbf{u}^{(2)}$, $\psi^{A(2)}$, it is possible to construct a singular solution due to a "center of rotation about the axis of z" [3]. We let the forces $h^{-1}P\mathbf{e}_x$ and $-h^{-1}P\mathbf{e}_y$ act at the origin (0, 0, 0), and the forces $-h^{-1}P\mathbf{e}_x$ and $h^{-1}P\mathbf{e}_y$ act at $(0, h, 0)$ and $(h, 0, 0)$, respectively, as shown in Fig. 1.

Passing to the limit as $h \to 0$ the resulting kinematical field is given by

$$\begin{aligned} \mathbf{u}^{(R)} &= \frac{\partial \mathbf{u}^{(1)}}{\partial y} - \frac{\partial \mathbf{u}^{(2)}}{\partial x} , \\ \psi^{A(R)} &= \frac{\partial \psi^{A(1)}}{\partial y} - \frac{\partial \psi^{A(2)}}{\partial x}. \end{aligned} \tag{11}$$

Computing the field $\mathbf{u}^{(R)}$ and $\psi^{A(R)}$ we obtain

$$\begin{aligned} \mathbf{u}^{(R)} &= -\frac{P}{4\pi\mu} (1 - l_3^2 \nabla^2) \mathbf{e}_z \times \nabla g_1 , \\ \psi^{A(R)} &= \frac{P}{8\pi\mu} \mathbf{I} \times \nabla \times \mathbf{e}_z \times \nabla g_1 . \end{aligned} \tag{12}$$

Expressions (12) may be obtained from (7) if we select

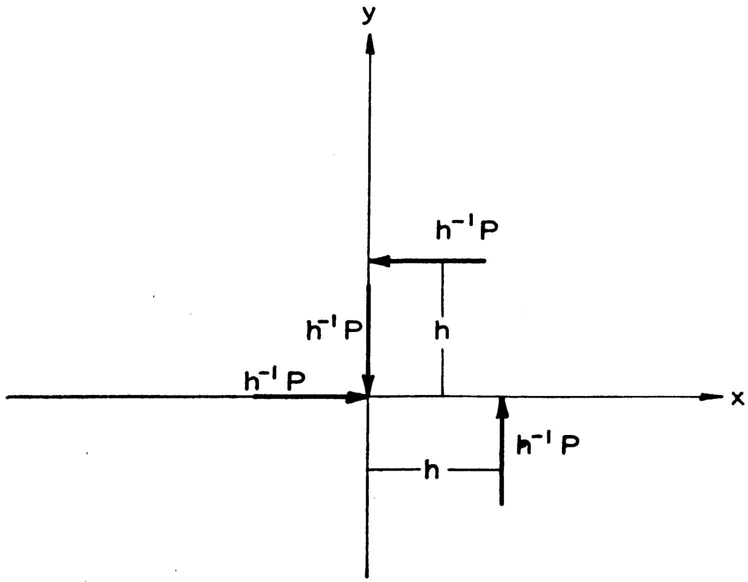

FIG. 1. The system of concentrated forces which yields, in the limit as $h \to 0$, a "center of rotation about the axis of z."

$$\mathbf{B} = \mathbf{B}^{(R)} = -\frac{P}{4\pi\mu}\mathbf{e}_z \times \nabla g_1, \qquad B_0^{(R)} = \mathbf{K}^{(R)} = K_0^{(R)} = 0. \tag{13}$$

For a concentrated moment directed about the z axis, at the origin $\mathbf{C} = C\mathbf{e}_z$, Eqs. (10) yield

$$\begin{aligned} \mathbf{B}^{(C)} &= -\frac{\mu+\beta}{8\pi\mu\beta}C\mathbf{e}_z \times \nabla g_1 = \frac{\mu+\beta}{2\beta}\frac{C}{P}\mathbf{B}^{(R)}, \\ B_0^{(C)} &= 0, \\ \mathbf{K}^{(C)} &= -\frac{C}{8\pi\beta r}\mathbf{e}_z, \\ K_0^{(C)} &= -\frac{l_2^2 C}{4\pi\beta}\frac{\partial g_2}{\partial z}. \end{aligned} \tag{14}$$

A comparison between (13) and (14) shows that the singularities due to the two kinds of concentrated couples are not the same. They differ by the stress functions $\mathbf{K}$ and K_0, given in (14). These functions yield a self-equilibrating kinematical field.

It may be worth noting that in the case of couple-stress theory [4], the singularity due to a concentrated couple and the singularity due to a "center of rotation" are the same.

3. Special cases. (a) Consider the following linear combination of the solutions to a concentrated couple $S^{(C)}$ and a center of rotation $S^{(R)}$.

$$S^{(L)} = -\frac{2P}{C}\frac{\beta}{\mu}S^{(C)} + \frac{\mu+\beta}{\mu}S^{(R)}. \tag{15}$$

Then

$$\begin{aligned}
\mathbf{B}^{(L)} &= -\frac{2P}{C}\frac{\beta}{\mu}\mathbf{B}^{(C)} + \frac{\mu+\beta}{\mu}\mathbf{B}^{(R)} = 0,\\
B_0^{(L)} &= 0,\\
\mathbf{K}^{(L)} &= -\frac{2P}{C}\frac{\beta}{\mu}\mathbf{K}^{(C)} = \frac{P}{4\pi\mu}\frac{\mathbf{e}_z}{r},\\
K_0^{(L)} &= -\frac{2P}{C}\frac{\beta}{\mu}K_0^{(C)} = \frac{Pl_2^2}{2\pi\mu}\frac{\partial g_2}{\partial z}.
\end{aligned} \tag{16}$$

The corresponding kinematical field is

$$\begin{aligned}
\mathbf{u}^{(L)} &= \frac{P}{4\pi\mu}\nabla\times\frac{\mathbf{e}_z}{r},\\
\boldsymbol{\psi}^{A(L)} &= \frac{P}{8\pi\mu}\mathbf{I}\times\nabla\frac{\partial g_2}{\partial z}.
\end{aligned} \tag{17}$$

It is interesting to note that the expression for $\mathbf{u}^{(L)}$ has thus been made to agree with the classical result for a center of rotation about the axis of z. It does not depend on the "micro-parameters" of the Cosserat medium.

(b) Consider the limit of the solution $S^{(C)}$, for a concentrated moment about the z axis, as the ratio $\mu/\beta \to \infty$.

In this case the characteristic lengths $l_1^2 \to l_3^2$ and

$$(1 - l_3^2\nabla^2)g_1 \to 1/r.$$

The kinematical fields become

$$\begin{aligned}
\mathbf{u}^{(C)} &\to 0,\\
\boldsymbol{\psi}^{A(C)} &= -\frac{C}{16\pi\beta}\mathbf{I}\times\left[\frac{e^{-r/l_1}}{rl_1^2}\mathbf{e}_z + \nabla\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_2}{\partial z}\right)\right].
\end{aligned} \tag{18}$$

It is seen that for this limiting case the macro-displacements vanish, and the resulting singular field contains micro-rotations alone.

Acknowledgments. The writer wishes to express his indebtedness to Professor R. D. Mindlin for a most valuable discussion.

This investigation was supported by the Office of Naval Research, Contract Nonr(G)00040-65, Task NR064-488, and by the Ordnance Research Laboratory at The Pennsylvania State University.

References

1. E. and F. Cosserat, *Théorie des corps deformables*, Hermann, Paris, 1909
2. R. D. Mindlin, *Stress functions for a Cosserat continuum*, Int. J. Solids Structures, **1**, 265–271 (1965)
3. A. E. H. Love, *A treatise on the mathematical theory of elasticity*, Dover, Fourth Ed. (Especially Sec. 132)
4. R. D. Mindlin and H. F. Tiersten, *Effects of couple-stresses in linear elasticity*, Arch. Rational Mech. Anal., **11**, 415–448 (1962)