MULTI-MODE SURFACE WAVE DIFFRACTION BY A RIGHT-ANGLED WEDGE*

BY

R. C. MORGAN**, S. N. KARP AND F. C. KARAL, JR.

(*Courant Institute of Mathematical Sciences, New York University*)

Abstract. This paper extends the phenomenological theory of multi-mode surface wave diffraction to a right-angled wedge configuration.

The solution to a two-mode problem is obtained under the edge condition

$$\sum_{j=0}^{2} \left| \frac{\partial^j u}{\partial x^j} \right| = 0(r^{-[1+h]}), \qquad 0 \leq h < \tfrac{2}{3}$$

as $r \to 0$. It is conjectured that the same procedure may be used to construct the solution to the corresponding N-mode problem under the edge condition

$$\sum_{j=0}^{N} \left| \frac{\partial^j u}{\partial x^j} \right| = 0(r^{-[(2N-1)/3+h]}), \qquad 0 \leq h \leq \tfrac{2}{3}$$

as $r \to 0$.

1. Introduction. This paper is concerned with extending the phenomenological theory of multi-mode surface wave diffraction proposed in Karp and Karal [1] and [2], where only plane structures were considered, to a right-angled wedge configuration (see Fig. 1). Previously, the solution to the single-mode problem was given by Karal and Karp [3] and uniqueness demonstrated in Morgan and Karp [4].

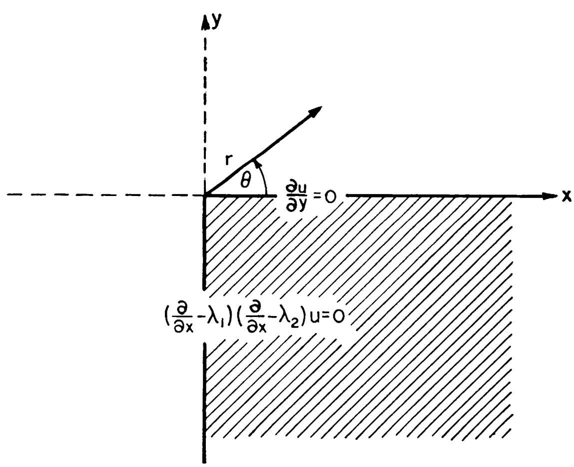

FIG. 1.

*Received November 5, 1965. The research reported in this paper was sponsored by the Air Force Cambridge Research Laboratories, Office of Aerospace Research under Contract No. AF 19(628)3868. The paper is based in part on R. C. Morgan's doctoral dissertation.

**Currently at Saint John's University, Jamaica, New York.

Here we consider a two-mode problem under a somewhat restrictive edge condition

$$\sum_{j=0}^{2} \left| \frac{\partial^j u}{\partial x^j} \right| = 0\left(\frac{1}{r^{1+h}}\right), \qquad 0 \le h < \tfrac{2}{3}$$

as $r \to 0$. The solution is obtained by extending the operator technique applied in [3].

The mathematical problem we wish to solve is posed by the following conditions:

i) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + K^2 u = 0, \qquad 0 \le \theta \le \tfrac{3}{2}\pi$

ii) $\left(\dfrac{\partial}{\partial x} - \lambda_1\right)\left(\dfrac{\partial}{\partial x} - \lambda_2\right)u = 0, \qquad \theta = \tfrac{3}{2}\pi$

$\dfrac{\partial u}{\partial y} = 0, \qquad \theta = 0$

where λ_1 and λ_2 are positive distinct constants.

iii) u and its derivatives satisfy the following conditions:

$$(a) \quad \sum_{j=0}^{3} \sum_{i=0}^{j} \left| \frac{\partial^j u}{\partial x^{j-i}\, \partial y^i} \right| < M \quad \text{for} \quad r > R_0 ,$$

where M is independent of r and θ and R_0 is some positive constant.

$$(b) \quad \sum_{i=0}^{2} \left| \frac{\partial^i u}{\partial x^i} \right| = 0\left(\frac{1}{r^{1+h}}\right)$$

as $r \to 0$ with $0 \le h < \tfrac{2}{3}$.

iv) $u = u_{\text{incident}} + u_{\text{reflected}} + u_{\text{radiated}}$,

where

$$u_{\text{inc.}} = \begin{cases} \displaystyle\sum_{m=1}^{2} A_m e^{+\lambda_m x + i\sqrt{K^2+\lambda_m^2}\, y}, & x < 0 \\ & y < 0 \\ 0, & y > 0 \end{cases}$$

and

$$u_{\text{refl.}} = \begin{cases} \displaystyle\sum_{m=1}^{2} B_m e^{+\lambda_m x - i\sqrt{K^2+\lambda_m^2}\, y}, & x < 0 \\ & y < 0 \\ 0. & y > 0. \end{cases}$$

Here A_m represents given incident surface wave amplitudes, and the B_m are constants representing reflection coefficients that must be determined.

v) $u_{\text{rad.}} \equiv u - u_{\text{inc.}} - u_{\text{refl.}}$ obeys the radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u_{\text{rad.}}}{\partial r} - iK u_{\text{rad.}} \right) = 0,$$

uniformly in θ, $0 \le \theta \le \tfrac{3}{2}\pi$ and vanishes at infinity.

2. Derivation of Solution. As mentioned previously, we extend the method of (3) by introducing the operator

$$v(x, y) = \left(\frac{\partial}{\partial x} - \lambda_1\right)\left(\frac{\partial}{\partial x} - \lambda_2\right)u(x, y) \tag{1.1}$$

where $u(x, y)$ is to be a solution of the boundary value problem posed in the Introduction. Consequently, $v(x, y)$ is a solution of the reduced wave equation satisfying the homogeneous boundary conditions

$$\begin{aligned} \frac{\partial v}{\partial y} &= 0, \qquad \theta = 0 \\ v &= 0, \qquad \theta = \tfrac{3}{2}\pi \end{aligned} \tag{1.2}$$

Hence, $v(x, y)$ may be developed in the cosine series

$$v(x, y) = \sum_{n=0}^{\infty} (D_n H^{(1)}_{2n+1/3}(\mathrm{Kr}) + E_n H^{(2)}_{2n+1/3}(\mathrm{Kr})) \cos \frac{2n+1}{3}\,\theta. \tag{1.3}$$

However, from the radiating behavior of $u_{\mathrm{rad.}}$, and the edge behavior of u, it follows that

$$v(x, y) = D_0 H^{(1)}_{1/3}(\mathrm{Kr}) \cos \frac{\theta}{3} + D_1 H^{(1)}_1(\mathrm{Kr}) \cos \theta. \tag{1.4}$$

Then, on inverting the operator v, we have

$$u(x, y) = \left\{ \begin{array}{ll} 0, \quad 0 \le \theta < \pi & \\ \sum_{m=1}^{2} A_m e^{+\lambda_m x + i\sqrt{K^2+\lambda_m^2}y} + B_m e^{+\lambda_m x - i\sqrt{K^2+\lambda_m^2}y} & \pi < \theta \le \dfrac{3\pi}{2} \\ \quad + \sum_{m=1}^{2} a_m e^{\lambda_m x} \int_x^{\infty} e^{-\lambda_m \xi} v(\xi, y)\, d\xi & \end{array} \right\} \tag{1.5}$$

where

$$a_m = \frac{(-1)^m}{\lambda_2 - \lambda_1}, \tag{1.6}$$

and $v(\xi, y)$ is given above (note x is replaced by ξ). Thus, we are left with the four unknown constants B_1, B_2, D_0, and D_1 which are obtainable by requiring the continuity of $u(x, y)$ and $\partial u(x, y)/\partial y$ across the negative x-axis. Consequently, on substituting the above form of $u(x, y)$ into the jump conditions

$$[u] = u(x, 0^+) - u(x, 0^-) = 0, \tag{1.7}$$

and

$$\left[\frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial y}(x, 0^+) - \frac{\partial u}{\partial y}(x, 0^-) = 0, \tag{1.8}$$

we arrive at the non-homogeneous system of four equations for the above four unknowns,

$$A_m = \tfrac{3}{2} a_m I(\tfrac{1}{3}, \lambda_m) D_0 - B_m, \tag{1.9}$$

$$\begin{aligned} i\sqrt{K^2 + \lambda_m^2}\, A_m = a_m \left\{ -K \sin \frac{4\pi}{3} e^{i2\pi/3} I(\tfrac{2}{3}, \lambda_m) - \lambda_m \sin \frac{2\pi}{3} I(\tfrac{1}{3}, \lambda_m) \right\} D_0 \\ - 4i\lambda_m a_m D_1 + i\sqrt{K^2 + \lambda_m^2}\, B_m, \end{aligned} \tag{1.10}$$

for $m = 1, 2$. Here, we define $I(\gamma, \lambda) = \int_0^{\infty} e^{-\lambda\xi} H^{(1)}_{\gamma}(K\xi)\, d\xi$ which is a known integral, see [5].

Finally, on solving the system, we obtain

$$D_0 = \frac{2i(\lambda_1 - \lambda_2)}{\Gamma} \{\lambda_2 \sqrt{K^2 + \lambda_1^2}\, A_1 + \lambda_1 \sqrt{K^2 + \lambda_2^2}\, A_2\}, \tag{1.11}$$

$$D_1 = \frac{1}{2} \frac{(\lambda_1 - \lambda_2)}{\lambda_1 \Gamma} \{(2\lambda_2 \sqrt{K^2 + \lambda_1^2}\, \alpha_1 - \sqrt{K^2 + \lambda_1^2}\, \lambda_1 \alpha_2) A_1 + \alpha_1 \lambda_1 \sqrt{K^2 + \lambda_2^2}\, A_2\}, \tag{1.12}$$

and

$$B_m = 3i(-1)^{m+1} \frac{I(\frac{1}{3}, \lambda_m)}{\Gamma} \{\lambda_2 \sqrt{K^2 + \lambda_1^2}\, A_1 + \lambda_1 \sqrt{K^2 + \lambda_2^2}\, A_2\} - A_m , \tag{1.13}$$

for $m = 1, 2$ where

$$\Gamma \equiv \lambda_2 \alpha_1 - \lambda_1 \alpha_2 , \tag{1.14}$$

and

$$\alpha_m \equiv -K \sin \frac{4\pi}{3} e^{i2\pi/3} I(\tfrac{2}{3}, \lambda_m) + \left(\tfrac{3}{2} i \sqrt{K^2 + \lambda_m^2} - \lambda_m \sin \frac{2\pi}{3}\right) I(\tfrac{1}{3}, \lambda_m). \tag{1.15}$$

In conclusion, we remark that it is evident that one could extend this method to higher order boundary conditions, that allow more surface waves, by simply increasing the order of the auxiliary operator $v(x, y)$ and restricting the edge behavior to be of the type

$$\sum_{\substack{i=0 \\ r \to 0}}^{N} \left| \frac{\partial^i u}{\partial x^i} (r, \theta) \right| = 0(r^{-[(2N-1)/3+h]}), \qquad 0 \le h < \tfrac{2}{3}, \tag{1.16}$$

where N is the number of surface wave modes supported. The difficulty will only exist in the algebra necessary to complete the solution.

Lastly, uniqueness has been demonstrated for the problem posed in (i)-(v) by Morgan [6]. Furthermore, it is conjectured that the solution to the N-mode problem, with the restriction (1.16) and a generalization of condition iii(a) (replace 3 by $N + 1$), will be unique.

Bibliography

1 F. C. Karal, and S. N. Karp, *Phenomenological Theory of Multi-Mode Surface Wave Excitation, Propagation and Diffraction*, I. Plane Structures, New York Univ., Courant Inst. Math. Sci., Div. of Electromagnetic Res., Res. Rep. No. EM–198, 1964

2. F. C. Karal, and S. N. Karp, *Phenomenological Theory of Multi-Mode Surface Wave Structures*, Quasi-Optics Symposium, Brooklyn Polytechnic Inst., (John Wiley and Sons, New York, 1964). Also, New York Univ., Courant Inst. Math. Sci., Div. of Electromagnetic Res., Res. Rep. No. EM–201, 1964

3. F. C. Karal, and S. N. Karp, *Scattering of a Surface Wave By a Discontinuity in the Surface Reactance on a Right-Angled Wedge*, Jointly with Chu, Ta-Shing, and Kouyoumjian, R. G., Comm. Pure and Appl. Math., 14, 1961, pp. 35–48. Also, New York Univ., Courant Inst. Math. Sci., Div. of Electromagnetic Res., Res. Rep. No. EM–146, 1960

4. R. C. Morgan, and S. N. Karp, *Uniqueness Theorem for a Surface Wave Problem in Electromagnetic Diffraction Theory*, Comm. Pure and Appl. Math., Vol. 16, 1963, pp 45–56. Also New York Univ., Courant Inst. Math. Sci., Div. of Electromagnetic Res., Res. Rep. No. EM–178, 1962.

5. W. Magnus, and F. Oberhettinger, Formulas and Theorems for the Special Functions of Mathematical Physics, 2nd Ed.; Berlin, Springer, 1948

6. R. C. Morgan, *Uniqueness Theorem for a Multi-Mode Surface Wave Problem in Electromagnetic Diffraction Theory*, New York Univ., Courant Inst. Math. Sci., Div. of Electromagnetic Res., Res. Rep. No. EM–212, 1965