A DISPLACEMENT POTENTIAL REPRESENTATION*

BY

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1. Introduction. In a recent paper, Mindlin [1] has obtained the displacement equilibrium equations for linear elastic medium in which the stresses are functions of the strains and the first and second strain gradients. It is the purpose of this note to establish the completeness of a displacement potential, for solutions of these equations, which is an extension of the well known Somigliana-Galerkin representation in classical elasticity. The potential itself satisfies a twelfth order partial differential equation. It may be resolved into a sum of six functions, each satisfying a lower order equation. In addition, a potential is given which may be used to generate solutions of higher order field equations.

2. Displacement potential. The field equations being studied are [1]:

$$b^2D_1^2D_2^2\nabla\nabla\cdot\mathbf{u} - a^2D_3^2D_4^2\nabla\times\nabla\times\mathbf{u} + \mathbf{F} = 0 \tag{2.1}$$

where a and b are the classical propagation speeds of equivoluminal and dilatation waves, $\mathbf{u}$ is the displacement vector, $\mathbf{F}$ is the body force per unit mass and the D operators are defined as follows:

$$D_i^2 \equiv 1 - l_i^2\nabla^2. \tag{2.2}$$

In this definition, the l_i are material constants. In what follows, all functions are assumed to have sufficient smoothness to permit all necessary operations to be performed.

Consider a general solution of the form:

$$\mathbf{u} = \frac{1}{b^2}D_3^2D_4^2\nabla\nabla\cdot\mathbf{g} - \frac{1}{a^2}D_1^2D_2^2\nabla\times\nabla\times\mathbf{g}. \tag{2.3}$$

Substitution of this displacement vector into the field equations (2.1) yields an equation for $\mathbf{g}$:

$$\nabla^4D_1^2D_2^2D_3^2D_4^2\mathbf{g} = -\mathbf{F}. \tag{2.4}$$

This is a twelfth order partial differential equation. Mindlin [1], on the other hand, developed a representation, resembling in structure the Popkovich-Neuber formula of classical elasticity, in which the vector and scalar potentials each satisfy a sixth order equation.

Utilizing a number of results from potential theory, one may show that a particular integral of Eq. (2.4) is given by:

$$\mathbf{g}(P) = \frac{1}{4\pi}\int_D \mathbf{F}(Q)\left\{\frac{r}{2} + \frac{A}{r} - \sum_{i=1}^{4} B_i l_i^{-2}\frac{\exp(-r/l_i)}{r}\right\}d\tau_Q \tag{2.5}$$

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where D is a regular region of space in which $\mathbf{u}$ is defined, and r is the distance from P to Q. The five constants A, B_i, must satisfy the following set of linear equations:

$$l_1^2 \equiv \alpha; \qquad l_2^2 \equiv \beta; \qquad l_3^2 \equiv \gamma; \qquad l_4^2 \equiv \delta$$

$$A = \alpha + \beta + \gamma + \delta$$

$$B_1 + B_2 + B_3 + B_4 = -A^2 - \alpha\beta - \alpha\gamma - \alpha\delta - \beta\gamma - \beta\delta - \gamma\delta$$

$$\begin{aligned}(\beta + \gamma + \delta)B_1 + (\alpha + \gamma + \delta)B_2 + (\alpha + \beta + \delta)B_3 + (\alpha + \beta + \gamma)B_4 \\ = \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta - A(\alpha\beta + \alpha\gamma + \beta\gamma)\end{aligned} \tag{2.6}$$

$$\begin{aligned}(\beta\gamma + \beta\delta + \gamma\delta)B_1 + (\alpha\gamma + \alpha\delta + \gamma\delta)B_2 + (\alpha\beta + \alpha\delta + \beta\delta)B_3 + (\alpha\beta + \alpha\gamma + \beta\gamma)B_4 \\ = -\alpha\beta\gamma\delta + A(\alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\gamma)\end{aligned}$$

$$\beta\gamma\delta B_1 + \alpha\gamma\delta B_2 + \alpha\beta\delta B_3 + \alpha\beta\gamma B_4 = A\alpha\beta\gamma\delta.$$

To verify the completeness of the representation described, define $\mathbf{g}$ as a combination of derivatives of a function $\mathbf{h}$:

$$\mathbf{g} = b^2 D_1^2 D_2^2 \nabla\nabla\cdot\mathbf{h} - a^2 D_3^2 D_4^2 \nabla \times \nabla \times \mathbf{h}. \tag{2.7}$$

$\mathbf{g}$ must satisfy Eq. (2.3), hence $\mathbf{h}$ is constrained to obey:

$$\nabla^4 D_1^2 D_2^2 D_3^2 D_4^2 \mathbf{h} = \mathbf{u}. \tag{2.8}$$

A particular integral of this equation follows immediately from Eq. (2.5). Therefore, for any given displacement $\mathbf{u}$ a vector $\mathbf{h}$ can be constructed; from Eq. (2.7) $\mathbf{g}$ is found corresponding to $\mathbf{h}$. This shows that the representation is complete.

The function $\mathbf{g}$ may be decomposed into six components obeying the following equations:

$$\begin{aligned} D_i^2 \mathbf{g}_i &= -B_i \mathbf{F}, \qquad i = 1, 2, 3, 4, \\ \nabla^4 \mathbf{g}_5 &= -\mathbf{F}, \\ \nabla^2 \mathbf{g}_6 &= -A\mathbf{F}. \end{aligned} \tag{2.9}$$

The various functions are defined in terms of combinations of derivatives of the resultant $\mathbf{g}$:

$$\begin{aligned} \mathbf{g}_1 &= B_1 \nabla^4 D_2^2 D_3^2 D_4^2 \mathbf{g}, \\ \mathbf{g}_2 &= B_2 \nabla^4 D_1^2 D_3^2 D_4^2 \mathbf{g}, \\ \mathbf{g}_3 &= B_3 \nabla^4 D_1^2 D_2^2 D_4^2 \mathbf{g}, \\ \mathbf{g}_4 &= B_4 \nabla^4 D_1^2 D_2^2 D_3^2 \mathbf{g}, \\ \mathbf{g}_5 &= D_1^2 D_2^2 D_3^2 D_4^2 \mathbf{g}, \\ \mathbf{g}_6 &= A \nabla^2 D_1^2 D_2^2 D_3^2 D_4^2 \mathbf{g}. \end{aligned} \tag{2.10}$$

It can be shown that when the constants A, B_i, satisfy Eqs. (2.6) the following holds:

$$\mathbf{g} = \sum_{i=1}^{6} \mathbf{g}_i . \tag{2.11}$$

In the event that $l_1 = l_2 = l_3 = 0$, $l_4 \neq 0$ the field equations reduce to the equations of couple-stress elasticity [2], furthermore the stress function representation given here reduces to that given in [3].

3. Extension to higher order equations. One might conjecture that if the constitutive relations are assumed to include third and higher strain gradients, the resulting equilibrium equations would be:

$$b^2D_1^2 \cdots D_n^2\nabla\nabla\cdot\mathbf{u} - a^2D_{n+1}^2 \cdots D_{2n}\nabla \times \nabla \times \mathbf{u} + \mathbf{F} = 0. \tag{3.1}$$

An argument in favor of such an extrapolation could be based on the fact that the equations should be invariant under coordinate transformations and since

$$\nabla \times \nabla \times \mathbf{u} = \nabla\nabla\cdot\mathbf{u} - \nabla^2\mathbf{u} \tag{3.2}$$

it follows that by proper choice of l_i in the operators of (3.1) any set of even order field equations could be cast in the form stated. For such a set of equations, a general solution similar to (2.3) is:

$$\mathbf{u} = \frac{1}{b^2} D_{n+1}^2 \cdots D_{2n}^2\nabla\nabla\cdot\mathbf{g} - \frac{1}{a^2} D_1^2 \cdots D_n^2\nabla \times \nabla \times \mathbf{g}. \tag{3.3}$$

In turn, the governing equation for $\mathbf{g}$ becomes:

$$\nabla^4D_1^2 \cdots D_{2n}^2\mathbf{g} = -\mathbf{F} \tag{3.4}$$

and finally, a particular integral for equation (3.4) is given by:

$$\mathbf{g}(P) = \frac{1}{4\pi}\int_D \mathbf{F}(Q)\left\{\frac{r}{2} + \frac{A}{r} - \sum_{i=1}^{2n} B_il_i^{-2}\frac{\exp(-r/l_i)}{r}\right\} d\tau_Q . \tag{3.5}$$

The constants are obtained as before, by equating coefficients of the various derivatives in the equation:

$$D_1^2 \cdots D_{2n}^2\mathbf{F} + A\nabla^2D_1^2 \cdots D_{2n}^2\mathbf{F} + B_1\nabla^4D_2^2 \cdots D_{2n}^2\mathbf{F} + \cdots + B_{2n}\nabla^4D_1^2 \cdots D_{2n-1}^2\mathbf{F} = \mathbf{F} \tag{3.6}$$

and solving the resulting set of $2n + 1$ linear equations. Proof of completeness and resolution of $\mathbf{g}$ into component functions is accomplished in exactly the same way as in the previous section.

References

1. R. D. Mindlin, *Second gradient of strain and surface-tension in linear elasticity,* Int. J. of Solids and Struct. **1**, (1965).
2. R. D. Mindlin and H. F. Tiersten, *Effects of couple-stresses in linear elasticity,* Arch. Rational Mech. Anal. **11**, (1962).
3. J. M. Doyle, *On completeness of stress functions in elasticity,* J. of Applied Mech. **31**, (1964).