QUARTERLY OF APPLIED MATHEMATICS

Vol. XXIV APRIL, 1966 No. 2

DYNAMIC BEHAVIOR OF SOAP FILMS*

BY

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1. Introduction. We consider a soap film stretched between two coaxial rings and we assume the potential energy of this surface to be proportional to the surface area. In this case we find [1] that the surface is rotationally symmetric and in fact takes the form of the catenary of revolution

$$r = \mu \cosh (x/\mu + \lambda), \tag{1.1}$$

as its position of stable equilibrium where x and r denote the distances along and from the axis of symmetry respectively; the two parameters λ and μ are chosen so as to satisfy the boundary conditions

$$r(0) = a, \qquad r(l) = b. \tag{1.2}$$

Assume the radii of the end rings (i.e. a and b) are fixed. In this case we find that there exists a certain critical value of l, say l_c, such that for $l > l_c$ there is no catenary of the form (1.1) connecting $(0, a)$ and (l, b). It may be verified, however, that for $l = l_c$ there is a catenary of the form (1.1) connecting $(0, a)$ and (l_c, b) and that this catenary does not touch the axis of symmetry. Experiments [1] indicate that when this critical point is reached the "throat" of the film shrinks rapidly until the film touches the axis of symmetry, at which point the film breaks. The object of this paper will be to describe the dynamic behavior of such films, and, in particular, to discuss the above experimental results from a dynamic viewpoint.

2. The Dynamic Equations. Let us assume that at some time $t = 0$ we parameterize the surface in terms of arc-length s. The initial surface will then be described by $x_0 = x_0(s)$ and $r_0 = r_0(s)$ where

$$\left(\frac{d}{ds} x_0\right)^2 + \left(\frac{d}{ds} r_0\right)^2 = 1. \tag{2.1}$$

If the end rings are moved apart or if the surface is not initially in a position of static equilibrium, the surface would be in motion at some time $t = \tau (\tau > 0)$. The position of any point a distance s on the initial surface will then be at a new position s at time $t = \tau$, so that we have $\mathrm{s} = \mathrm{s}(s, t)$. This deformed surface will be described by equations

*Received April 20, 1965; revised manuscript received November 15, 1965. This research was done at the Courant Institute of Mathematical Sciences, New York University, under the joint sponsorship of the I. B. M. System Research Institute, and the National Aeronautics and Space Administration under Contract No. NsG–412.

$x = x(\mathbf{s}) = x(s, t)$ and $r = r(\mathbf{s}) = r(s, t)$. Assuming the potential energy to be proportional to the surface area we may write the potential energy as

$$V = 2\pi T \int_0^{s_f} r(x_s^2 + r_s^2)^{1/2}\, ds, \tag{2.2}$$

where T is some proportionality constant and we assume axial symmetry is retained. Assuming mass is conserved the kinetic energy becomes

$$U = 2\pi \int_0^{s_f} \frac{\rho}{2} r_0(x_t^2 + r_t^2)\, ds, \tag{2.3}$$

where ρ is the initial surface density (assumed constant). Therefore Hamilton's principle[1] requires that the surface move so that

$$2\pi \int_0^{\tau} \int_0^{s_f} (\tfrac{1}{2}\rho r_0(r_t^2 + x_t^2) - \mathrm{Tr}\,(x_s^2 + r_s^2)^{1/2})\, ds\, dt \tag{2.4}$$

is stationary. From (2.4) we find the equations of motion to be

$$\rho r_0 x_{tt} = T \frac{\partial}{\partial s} \frac{r x_s}{(x_s^2 + r_s^2)^{1/2}} \tag{2.5a}$$

$$\rho r_0 r_{tt} = T \frac{\partial}{\partial s} \frac{r r_s}{(x_s^2 + r_s^2)^{1/2}} - T(x_s^2 + r_s^2)^{1/2}. \tag{2.5b}$$

On physical grounds it would be reasonable to prescribe the initial-boundary value problem defined by

$$x(s, 0) = x_0(s), \qquad r(s, 0) = r_0(s), \tag{2.6a}$$

$$x_t(s, 0) = 0, \qquad r_t(s, 0) = 0, \tag{2.6b}$$

and

$$x(0, t) = f_0(t), \qquad x(s_f\,, t) = f_1(t), \tag{2.7a}$$

$$r(0, t) = a, \qquad r(s_f\,, t) = b,$$

where s_f is the initial total meridional length. Of course we assume the initial conditions (2.6a), (2.6b) and the boundary conditions (2.7a), (2.7b) satisfy the required continuity conditions.

3. Discontinuities. We may write down the characteristic determinant [4] of (2.5a) and (2.5b) from which we find that the characteristic curve $\phi(s, t) = \text{constant}$ must satisfy

$$\phi_t^2\left(\phi_t^2 - \frac{\mathrm{Tr}}{\rho r_0(x_s^2 + r_s^2)^{1/2}}\, \phi_s^2\right) = 0. \tag{3.1}$$

Hence the characteristic slopes are given by

$$\frac{ds}{dt} = 0, \tag{3.2}$$

[1]These equations may also be derived from Newton's second law of motion if we assume the surface to be a shear-free elastic surface supporting a constant tension per unit length T [3].

and

$$(ds/dt)^2 = \frac{\mathrm{Tr}}{\rho r_0(x_s^2 + r_s^2)^{1/2}}. \tag{3.3}$$

We see that we have four real characteristic directions, two of which are identical and independent of the solution; thus the equations (2.5) are hyperbolic.

The equation (3.2) indicates that we might expect difficulties with the initial boundary value problem since the boundary $s = 0$ and $s = s_f$ are characteristics and hence the boundary data is characteristic data. In order to exhibit the difficulty we write (2.5a) and (2.5b) as a second order system

$$\rho r_0 \begin{bmatrix} x \\ r \end{bmatrix}_{tt} = \frac{\mathrm{Tr}}{(x_s^2 + r_s^2)^{3/2}} \begin{bmatrix} -r_s^2 & x_s r_s \\ x_s r_s & -x_s^2 \end{bmatrix} \begin{bmatrix} x \\ r \end{bmatrix}_{ss} - \frac{Tx_s}{(x_s^2 + r_s^2)^{1/2}} \begin{bmatrix} -r_s \\ x_s \end{bmatrix}. \tag{3.4}$$

If one multiplies the equation (3.4) from the left by the row vectors (x_s, r_s) and $(r_s, -x_s)$ we obtain the new system of equations

$$x_s x_{tt} + r_s r_{tt} = 0 \tag{3.5a}$$

$$r_s x_{tt} - x_s r_{tt} = \frac{T(x_s^2 + r_s^2)}{\rho r_0 r_s} \frac{\partial}{\partial s} \frac{r x_s}{(x_s^2 + r_s^2)^{1/2}}. \tag{3.5b}$$

We recognize the left side of (3.5a) as the component of acceleration tangential to the surface. Therefore we see from (3.5a) that the acceleration is always normal to the surface.

Consider the initial boundary value problem prescribed by (2.6) and (2.7) (e.g. the catenary). If we accelerate one end smoothly from rest then at $t = \epsilon > 0$ we have $x_{tt} \neq 0$. However, the radii of the end surfaces $r(0, t)$ and $r(s_f, t)$ are fixed for all time corresponding to the physical situation of the surface being attached to rings of these radii. We see then that at the ends $r_{tt} = 0$ for all time. Combining the above remarks with (3.5a) we see that at $t = \epsilon$, $x_s = 0$. For surfaces such as the catenary of revolution $x_s \neq 0$ at $t = 0$. Hence we conclude that $x_s(s_f, t)$ cannot be continuous at time $t = 0$ if these components are moved from rest.

4. Solutions for Small Time. We shall find it convenient to develop a linear system of equations that is valid for small time, and which can be *solved explicitly*. In fact we will see later that the solution is a good approximation for the solution of the nonlinear equations up to time $t = 1$!

If we write (2.5a) and (2.5b) as a first order system we easily convince ourselves that the coefficient matrices are analytic in a neighborhood of $t = 0$. Let us assume that the initial data is also analytic in some neighborhood of $t = 0$. Let s be any point such that $0 < s < s_f$, then by the Cauchy–Kowalewski theorem [2] we know that there exists an analytic solution in some neighborhood of $(s, 0)$. Since this is true for any s such that $0 < s < s_f$ we may easily guarantee the existence of an analytic solution in some "lens shaped" region $t = t^+(s)$ where $0 < s < s_f$ and $t^+(0) = t^+(s_f) = 0$. Let us rewrite equations (2.5a) and (2.5b) as

$$r_0 x_{tt} - \frac{\partial}{\partial s} r_0 \psi x_s = 0, \tag{4.1a}$$

$$r_0 r_{tt} - \frac{\partial}{\partial s} r_0 \psi r_s + \frac{T^2}{\rho^2 r_0 \psi} = 0, \tag{4.1b}$$

where ψ is the square of the "sound speeds" (Cf. 3.3):

$$\psi = \frac{\mathrm{Tr}}{\rho r_0 (x_s^2 + r_s^2)^{1/2}}. \tag{4.2}$$

ψ is an analytic function of the solution and the initial data. Therefore we would expect to be able to write ψ as

$$\psi(s, t) = \sum_{n=0}^{n=\infty} \sigma_n(s) t^n, \tag{4.3}$$

convergent in some lens shaped region, where the coefficients σ_n are analytic functions of s. Let us evaluate the σ_n's from the initial data and the differential equations. We have

$$\sigma_0 = \psi(s, 0) = \frac{\mathrm{Tr}\,(s, 0)}{\rho r_0 (x_s(s, 0)^2 + r_s(s, 0)^2)^{1/2}}, \tag{4.4}$$

or

$$\sigma_0 = \frac{T}{\rho}. \tag{4.5}$$

We may also show that if $x_t(s, 0) = r_t(s, 0) = 0$ then

$$\sigma_1 = 0, \tag{4.6}$$

so that we may write

$$\psi(s, t) = \frac{T}{\rho} + \sum_{n=2}^{n=\infty} \sigma_n(s) t^n. \tag{4.7}$$

Therefore for t small we see

$$\psi(s, t) \approx \frac{T}{\rho}. \tag{4.8}$$

Replacing ψ in equations (4.1a) and (4.1b) by (4.8) we obtain

$$x_{tt} - \frac{T}{\rho} x_{ss} = 0, \tag{4.9a}$$

$$r_{tt} - \frac{T}{\rho} r_{ss} + \frac{T}{\rho r_0^2} r = 0, \tag{4.9b}$$

and we would expect these equations to be valid for small time. We will solve equations (4.9a) and (4.9b) explicitly in the next section.

5. Non-Linear and Small Time Solutions. Consider the initial data

$$x(s, 0) = s, \qquad x_t(s, 0) = 0, \tag{5.1a}$$

$$r(s, 0) = a, \qquad r_t(s, 0) = 0, \tag{5.1b}$$

and the boundary data

$$x(0, t) = 0, \qquad x(s_f, t) = s_f, \tag{5.2a}$$

$$r(0, t) = r(s_f, t) = a, \tag{5.2b}$$

i.e. we consider a soap film which is in the form of a cylinder of radius a at time $t = 0$. This is not a position of static equilibrium for the soap film so that the film is in motion at some later time. We will solve equations (4.9a) and (4.9b) for this initial boundary value problem. The solution of (4.9a) is given by

$$x = s. \tag{5.3}$$

In order to solve (4.9b) introduce

$$r(s, t) = Z(s, t) + \zeta(s) + a \tag{5.4}$$

where

$$\zeta(s) = Ae^{s/a} + Be^{-s/a} - a \tag{5.5}$$

and

$$A = \frac{a(1 - e^{-s_f/a})}{2 \sinh (s_f/a)}, \tag{5.6}$$

$$B = \frac{a(e^{s_f/a} - 1)}{2 \sinh (s_f/a)}. \tag{5.7}$$

Equation (4.9b) becomes

$$Z_{tt} - \frac{T}{\rho} Z_{ss} + \frac{T}{\rho a^2} Z = 0, \tag{5.8}$$

and Z satisfies initial conditions

$$Z(s, 0) = -\zeta(s), \qquad Z_t(s, 0) = 0, \tag{5.9}$$

and boundary conditions

$$Z(0, t) = Z(s_f, t) = 0. \tag{5.10}$$

We may solve (5.8) in the form of a uniformly convergent Fourier series, so that we find

$$r = Ae^{s/a} + Be^{-s/a} + \frac{4s_f^2}{a\pi} \sum_{n=0}^{n=\infty} \frac{\sin \left(\frac{2n+1}{s_f} \pi s\right) \cos (\lambda_n t)}{(2n+1)((2n+1)^2\pi^2 + s_f^2/a^2)} \tag{5.11}$$

where

$$\lambda_n = \left(\frac{T}{\rho} \left(\frac{(2n+1)^2\pi^2}{s_f^2} + \frac{1}{a^2}\right)\right)^{1/2}. \tag{5.12}$$

The non-linear equations (2.5a) and (2.5b) with initial conditions (5.1a), (5.1b) and boundary conditions (5.2a), (5.2b) were solved numerically by replacing derivatives by centered difference quotients. The solution is plotted in Figure 1. For these solutions T/ρ was chosen as 1. However, the actual numerical value of T/ρ affects only the speed at which the film moves and not its form. The separation of the end rings was chosen as 1.3255, which is very close to the critical distance at which there exists no static solution. We see in Figure 1 that the film passes through the equilibrium position (the

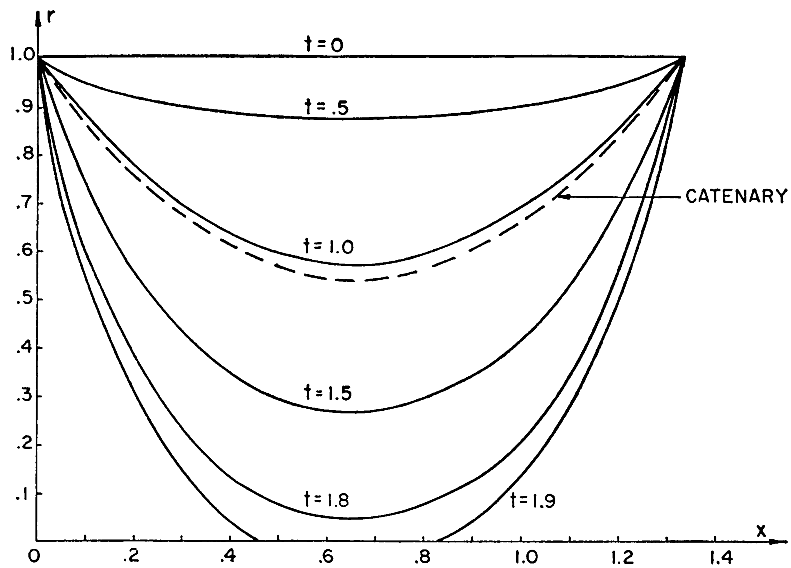

FIG. 1.

catenary)[1] and eventually touches the axis. At this point the film would break and the equations (2.5a) and (2.5b) would no longer describe the physical situation. In Figure 2 the solution of the non-linear equations is compared with the solution (5.11) of the "small time" equations (4.9a) and (4.9b). We see that for $t \leq 1$ the approximation is quite good. For $t > 1$ the solution (5.11) no longer furnishes a useful approximation to the actual solution.

In the introduction it was indicated that if the distance between the end rings exceeded a certain critical value the solution to the static problem ceased to exist. It is of interest to investigate what happens dynamically when this value is exceeded. For this purpose the equations (4.9a) and (4.9b) were solved numerically with the catenary of revolution as the initial surface under boundary conditions

$$x(0, t) = 0, \qquad x(s_f, t) = 1.3255 + .05t^3, \tag{5.13a}$$

$$r(0, t) = r(s_f, t) = 1 \tag{5.13b}$$

and a second set

$$x(0, t) = 0, \qquad x(s_f, t) = \begin{cases} 1.3255, & 0 \leq t \\ 1.3255 + 5.235t^3\left(\dfrac{t^2}{5} - \dfrac{t}{2} + \dfrac{1}{3}\right), & 0 \leq t \leq 1 \\ 1.5, & 1 \leq t \end{cases} \tag{5.14a}$$

$$r(0, t) = r(s_f, t) = 1. \tag{5.14b}$$

In Figure 3 we have plotted the solution for the boundary conditions (5.13a) and (5.13b). This corresponds to the situation of accelerating the right end ring smoothly from

[1]These exist two catenaries satisfying the prescribed boundary conditions. However, only the catenary shown in Fig. 1 yields a minimal value for the energy (Cf. [1]).

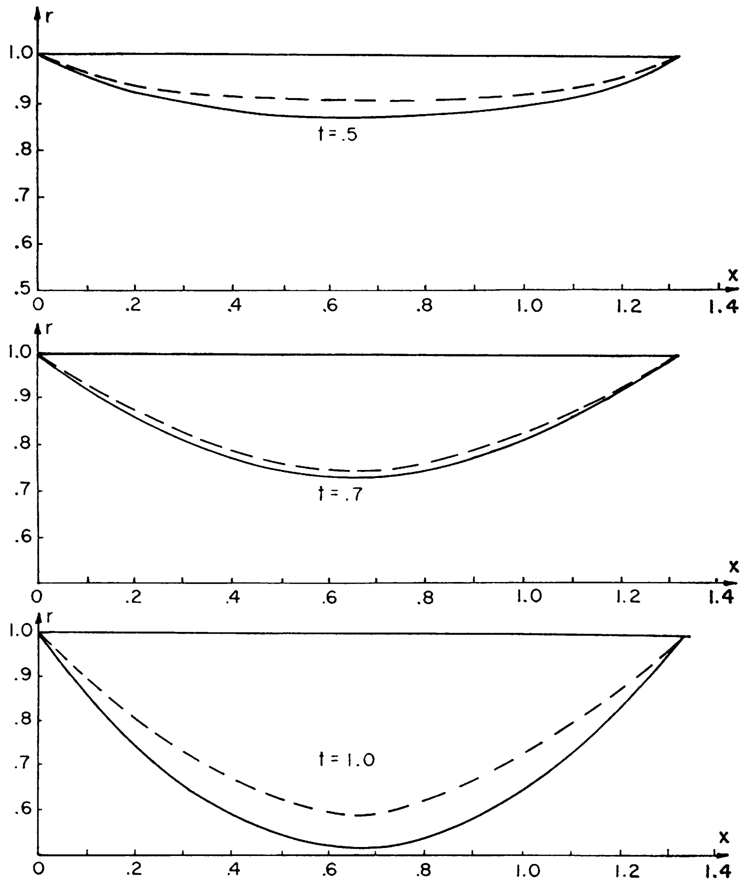
r
1.0
.9
.8
.7
.6
.5
t = .5
t = .7
t = 1.0
x
0
.2
.4
.6
.8
1.0
1.2
1.4

Fig. 2.

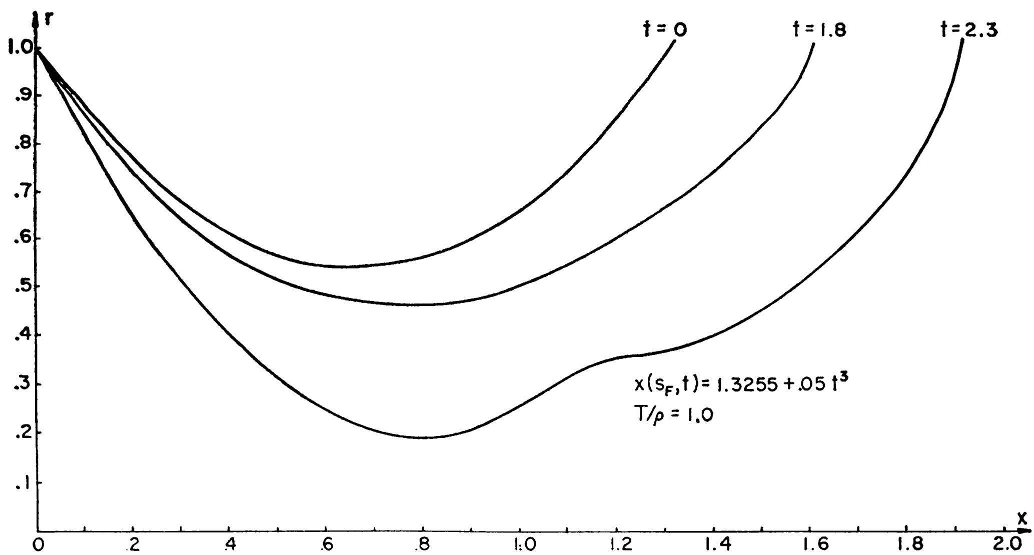
r
t = 0
t = 1.8
t = 2.3
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$x(s_F, t) = 1.3255 + .05\,t^3$
$T/\rho = 1.0$
x
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0

Fig. 3.

rest through the critical position where the static solution ceases to exist. One sees that at $t = 2.3$ a noticeable "hump" has occurred in the film. Observations made on other boundary value solutions indicate that this occurs when the velocity of the end rings approaches the "sound speeds" of the equations (2.5a) and (2.5b) and it is probable that this phenomenon is related to the fact that the acceleration must be normal to the surface. Figure 4 is a graph of the solution to the problem with boundary conditions (5.14a) and (5.14b). This corresponds to the situation of moving the end ring smoothly from rest through the position where the static solution ceases to exist and then bringing the end ring smoothly to rest. The bulge in the film which occurs at the end ring may be interpreted as an inertia effect. We see that after the end ring is brought to rest, the film continues to contract at the center until the axis is touched. Consequently the phenomenon of the "throat" radius shrinking to zero mentioned in Section 1 does seem to be contained in the equations (2.5a) and (2.5b). The case where the end rings exceed the critical distance is not the only situation where the throat radius contracts to zero (Cf. Fig. 1). This result depends on 1) the distance of the initial surface from the position of static equilibrium and 2) the distance between the end rings.

6. The Sphere. If we attempt to find a solution of (2.5*a*) and (2.5*b*) of the form $x = X(s)f(t)$, $r = R(s)f(t)$ and simultaneously require $x(s, 0) = x_0(s)$, $r(s, 0) = r_0(s)$, we find $X(s) = x_0(s)$ and $R(s) = r_0(s)$ or

$$x(s, t) = x_0(s)f(t), \tag{6.1a}$$

$$r(s, t) = r_0(s)f(t). \tag{6.1b}$$

Noting that $\partial \mathbf{s}/\partial s = (x_s^2 + r_s^2)^{1/2}$ it is easily found that a solution of the form (6.1a) and (6.1b) requires that $\mathbf{s}$ be in a linear relationship to s i.e.

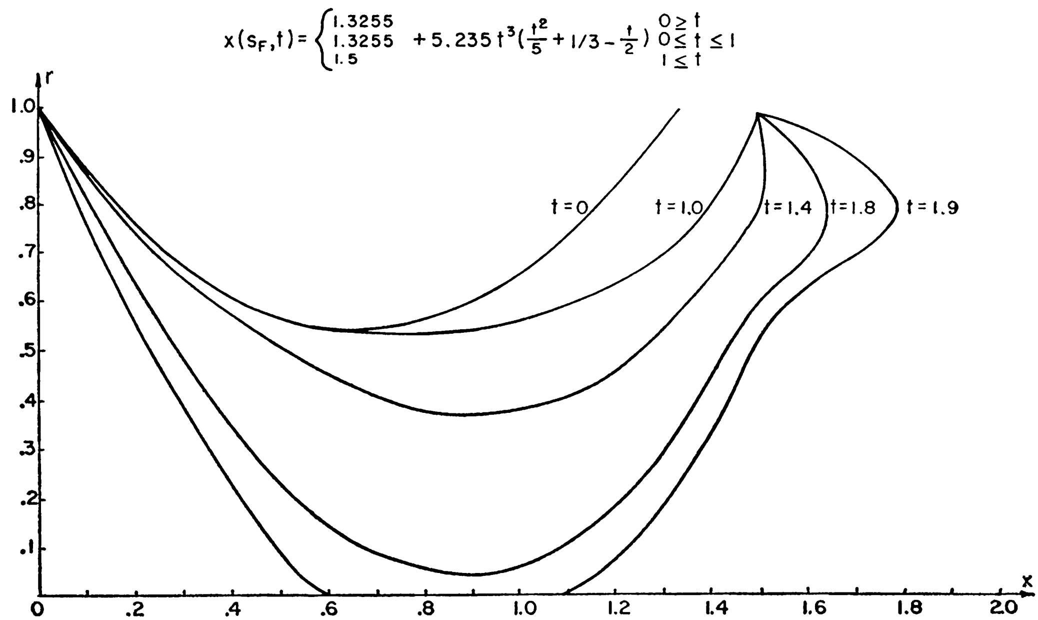

Fig. 4.

$$\mathrm{s} = f(t)s. \tag{6.2}$$

Substituting (6.1a) and (6.1b) into (2.5a) and (2.5b) we find that the variables separate to yield

$$\frac{d}{ds}\left(r_0 \frac{dx_0}{ds}\right) + \frac{\alpha\rho}{T} r_0 x_0 = 0, \tag{6.3a}$$

$$\frac{d}{ds}\left(r_0 \frac{dr_0}{ds}\right) + \frac{\alpha\rho}{T} r_0^2 = 0, \tag{6.3b}$$

and

$$\frac{d^2 f}{dt^2} - \alpha f = 0, \tag{6.4}$$

for some constant α. For $\alpha = 0$ the solution of (6.3a) and (6.3b) is the catenary. However, from (6.4) with $\alpha = 0$ we see that the only dynamic solution is the case of uniform contraction or expansion (including expansion or contraction of the end rings). A more interesting case occurs if $\alpha \neq 0$. Performing the differentiations in equations (6.3a) and (6.3b), multiplying (6.3a) by x_{0s} and (6.3b) by r_{0s} , and adding we find

$$r_{0s}(x_{0s}^2 + r_{0s}^2) + r_0(x_{0s}x_{0ss} + r_{0s}r_{0ss}) + \alpha r_0 x_0 x_{0s} + \alpha r_0^2 r_{0s} - r_{0s} = 0. \tag{6.6}$$

Recalling that $x_{0s}^2 + r_{0s}^2 = 1$ and hence $x_{0s}x_{0ss} + r_{0s}r_{0ss} = 0$ we find

$$x_0^2 + r_0^2 = a^2. \tag{6.7}$$

Therefore the spherical initial surface should yield a solution of the form (6.1a) and (6.1b). Noting that the sphere of radius a may be parameterized as $x_0 = -a \cos(s/a)$, $r_0 = a \sin(s/a)$ we make the substitution $x = -a \cos(s/a)f(t)$, $r = a \sin(s/a)f(t)$ into equations (2.5a) and (2.5b). We find that both equations reduce to the same ordinary differential equation for f,

$$\rho a^2 \frac{d^2 f}{dt^2} + 2Tf = 0. \tag{6.8}$$

If we solve (6.7) with initial conditions $f(0) = 1$, $f'(0) = 0$ we find the solution for the sphere to be

$$x = -a \cos\left(\frac{s}{a}\right) \cos \frac{1}{a}\left(\frac{2T}{\rho}\right)^{1/2} t, \tag{6.9a}$$

$$r = a \sin\left(\frac{s}{a}\right) \cos \frac{1}{a}\left(\frac{2T}{\rho}\right)^{1/2} t. \tag{6.9b}$$

From (6.9a) and (6.9b) we find the time t_c required for the soap bubble to collapse to a point to be

$$t_c = \frac{\pi a}{4}\left(\frac{2\rho}{T}\right)^{1/2}. \tag{6.10}$$

Acknowledgements. The author is indebted to J. J. Stoker who acted as his research advisor during the preparation of the Ph.D. thesis from which this material is taken, and to E. Isaacson for his suggestions during the preparation of this paper.

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