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A NOTE ON A WORK INEQUALITY IN LINEAR VISCOELASTICITY*

By J. B. MARTIN AND A. R. S. PONTER** (*Brown University*)

In recent years various writers have studied the development and application of work or quasi-thermodynamic inequalities for common idealizations of material behavior. For time-dependent materials, a postulate of material stability was introduced by Drucker [1] as an extension of similar work for time-independent materials [2]. This postulate defines a class of materials for which the solutions to standard boundary value problems are unique. In linear viscoelasticity, Breuer and Onat [3] have used similar ideas to determine the restrictions which must be placed on the relaxation function in order to guarantee uniqueness in the same sense. Breuer and Onat assumed that the work required to deform a linear viscoelastic material from the virgin state is non-negative. This statement, together with the linearity, can be used for a uniqueness proof.

In this note we consider an isotropic linear viscoelastic solid whose mechanical behavior can be represented by a generalization of a model composed of an infinite number of interconnected springs and dashpots. We assume that in the time interval $(-\infty, 0)$ the solid is in the unstressed and unstrained virgin state. We consider the specific work required to deform the solid, and we wish to show that the specific work can be bounded from below by a non-negative expression which depends only on the strain at time t and time t, and not on the history.

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote the components of the stress and infinitesimal strain tensors. Let $\mathbf{s}$ and $\mathbf{e}$ denote the stress and strain deviators and let p, v denote hydrostatic tension and isotropic volume strain. If $J_1(t)$ and $J_2(t)$ are the creep functions of the material in pure shear and dilatation respectively, the constitutive equations may be written as

$$\mathbf{e} = \int_0^t J_1(t - \tau)\dot{\mathbf{s}}(\tau)\, d\tau \tag{1a}$$

and

$$v = \int_0^t J_2(t - \tau)\dot{p}(\tau)\, d\tau. \tag{1b}$$

This representation assumes that the material is isotropic and that the material is unstressed and unstrained for $t < 0$. We shall restrict our attention to particular forms of J_1 and J_2. To introduce these forms we consider first the shear deformation. We assume that the strain deviator $\mathbf{e}$ can be written as the sum of the strains in n elements of a model.

$$\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n . \tag{2}$$

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**Present address: University of Glasgow.

The number of elements in the model can be as large as is required, and can be infinite. The stress deviator associated with each element of the model is $\mathbf{s}$. We assume that the jth element of the model is governed by the differential equation

$$E_j\mathbf{e}_j + \nu_j\dot{\mathbf{e}}_j = \mathbf{s} \tag{3}$$

where E_j , ν_j are non-negative constant coefficients. If the element is subjected to a step function in stress, denoted by $\mathbf{s}H(t)$ with $\mathbf{s}$ constant, the solution to (3) is given by

$$\mathbf{e}_j = \frac{\mathbf{s}}{E_j}(1 - e^{-E_j/\nu_j t}). \tag{4}$$

It follows therefore that the creep function is given by:

$$J_1(t) = \sum_{j=1}^{n} \frac{1}{E_j}(1 - e^{-E_j/\nu_j t}). \tag{5}$$

In the case of simple shear, where the stress and strain deviators have only one non-zero component, this representation is equivalent to a mechanical model of the material behavior consisting of n Voigt elements connected in series. The restriction placed on the creep function is simply that it must be possible to express it in terms of the finite of infinite sum given in (5).

The dilatation v is similarly divided into n parts, and each element of the model is assumed to satisfy the differential equation

$$K_jv_j + \mu_j\dot{v}_j = p \tag{6}$$

where K_j , μ_j are non-negative constant coefficients. The creep function is then given by

$$J_2(t) = \sum_{j=1}^{n} \frac{1}{K_j}(1 - e^{-K_j/\mu_j t}). \tag{7}$$

We return now to a consideration of shear behavior. It is convenient to introduce two stress tensors in each element of the model defined by

$$\mathbf{q}_j = E_j\mathbf{e}_j \tag{8a}$$

$$\mathbf{r}_j = \nu_j\dot{\mathbf{e}}_j . \tag{8b}$$

Thus, from (3),

$$\mathbf{q}_j + \mathbf{r}_j = \mathbf{s}. \tag{9}$$

We next introduce arbitrary constant deviatoric stress tensors $\mathbf{q}_j^*$ and $\mathbf{r}_j^*$. Noting that E_j , ν_j are non-negative, the stability of the elastic and viscous relations (8a) and (8b) may be expressed by means of a Schwarz inequality:

$$\frac{1}{E_j}(\mathbf{q}_j^* - \mathbf{q}_j)\cdot(\mathbf{q}_j^* - \mathbf{q}_j) \geq 0 \tag{10a}$$

$$\frac{1}{\nu_j}(\mathbf{r}_j^* - \mathbf{r}_j)\cdot(\mathbf{r}_j^* - \mathbf{r}_j) \geq 0. \tag{10b}$$

Consistent with the assumption of an initial virgin state, $\mathbf{e}_j$ is taken to be zero at time $t = 0$. Using (8b), and integrating over the interval $(0, t)$, (10b) becomes:

$$\frac{\mathbf{r}_j^*\cdot\mathbf{r}_j^*}{2\nu_j}\,t + \frac{1}{2}\int_0^t \mathbf{r}_j(\tau)\cdot\dot{\mathbf{e}}_j(\tau)\,d\tau \geq \mathbf{r}_j^*\mathbf{e}_j\ . \tag{11}$$

Using (8a), (10a) may be written as

$$\frac{\mathbf{q}_j^*\cdot\mathbf{q}_j^*}{2E_j} + \int_0^t \mathbf{q}_j(\tau)\cdot\dot{\mathbf{e}}_j(\tau)\,d\tau \geq \mathbf{q}_j^*\cdot\mathbf{e}_j\ . \tag{12}$$

Now we consider the specific work done in shear deformation of the solid in the interval $(0, t)$ which is given by the expression

$$W_s = \int_0^t \mathbf{s}\cdot\dot{\mathbf{e}}\,d\tau. \tag{13}$$

By using (2) and (9), (13) can be written in the following form

$$W_s = \left\{\int_0^t \mathbf{q}_1\cdot\dot{\mathbf{e}}_1\,d\tau + \int_0^t \mathbf{q}_2\cdot\dot{\mathbf{e}}_2\,d\tau + \cdots + \int_0^t \mathbf{q}_n\cdot\dot{\mathbf{e}}_n\,d\tau\right\} + 2\left\{\frac{1}{2}\int_0^t \mathbf{r}_1\cdot\dot{\mathbf{e}}_1\,d\tau + \frac{1}{2}\int_0^t \mathbf{r}_2\cdot\dot{\mathbf{e}}_2\,d\tau + \cdots + \frac{1}{2}\int_0^t \mathbf{r}_n\cdot\dot{\mathbf{e}}_n\,d\tau\right\}. \tag{14}$$

It can then be seen that by repeated applications of (11) and (12), W_s can be bounded from below:

$$W_s \geq \sum_{j=1}^n (\mathbf{q}_j^* + 2\mathbf{r}_j^*)\cdot\mathbf{e}_j - \sum_{j=1}^n \left\{\frac{\mathbf{q}_j^*\cdot\mathbf{q}_j^*}{2E_j} + t\,\frac{\mathbf{r}_j^*\cdot\mathbf{r}_j^*}{\nu_j}\right\} \tag{15}$$

where $\mathbf{q}_j^*$ and $\mathbf{r}_j^*$ are arbitrary tensor valued constants. We now restrict $\mathbf{q}_j^*$ and $\mathbf{r}_j^*$ by requiring that

$$\mathbf{q}_j^* + 2\mathbf{r}_j^* = \mathbf{s}^*. \tag{16}$$

Thus

$$\sum_{j=1}^n (\mathbf{q}_j^* + 2\mathbf{r}_j^*)\cdot\mathbf{e}_j = \mathbf{s}^*\cdot\mathbf{e}. \tag{17}$$

Further, we put

$$\mathbf{q}_j^* = m_j\mathbf{s}^*, \qquad \mathbf{r}_j^* = \left(\frac{1 - m_j}{2}\right)\mathbf{s}^*, \tag{18}$$

m_j may still be arbitrarily chosen. In order to obtain the best bound on W_s we choose m_j such that

$$\frac{\mathbf{q}_j^*\cdot\mathbf{q}_j^*}{2E_j} + t\,\frac{\mathbf{r}_j^*\cdot\mathbf{r}_j^*}{\nu_j}\,,$$

has a minimum value. It is easily shown that this occurs when

$$m_j = \frac{tE_j}{tE_j + 2\nu_j}. \tag{19}$$

Then

$$\frac{\mathbf{q}_j^*\cdot\mathbf{q}_j^*}{2E_j} + t\,\frac{\mathbf{r}_j^*\cdot\mathbf{r}_j^*}{\nu_j} = \frac{\mathbf{s}^*\cdot\mathbf{s}^*}{2}\cdot\frac{t}{(tE_j + 2\nu_j)}. \tag{20}$$

(15) becomes

$$W_s \geq \mathbf{s}^* \cdot \mathbf{e} - \frac{\mathbf{s}^* \cdot \mathbf{s}^*}{2} \sum_{j=1}^{n} \frac{t}{(tE_j + 2\nu_j)} \tag{21}$$

Finally we choose $\mathbf{s}^*$ such that the right hand side of (21) is a maximum. This is achieved by considering in turn each component of $\mathbf{s}^*$ and the associated component of $\mathbf{e}$. It is readily shown that the optimum value of $\mathbf{s}^*$ is given by

$$\mathbf{s}^* = \frac{\mathbf{e}(t)}{\displaystyle\sum_{j=1}^{n} \frac{t}{tE_j + 2\nu_j}} \tag{22}$$

and the best bound on W_s is hence

$$W_s \geq \frac{\mathbf{e} \cdot \mathbf{e}}{\displaystyle\sum_{j=1}^{n} \frac{2t}{tE_j + 2\nu_j}}. \tag{23}$$

It is convenient to express this result in terms of the creep function $J_1(t)$. The Laplace transform of $J_1(t)$ given in (5) is

$$\bar{J}_1(p) = \int_0^\infty e^{-pt} J_1(t)\, dt = \sum_{j=1}^{n} \frac{1}{E_j} \frac{E_i/\nu_i}{p(p + E_j/\nu_j)}. \tag{24}$$

It is readily seen that (23) may then be written as

$$W_s \geq \frac{\mathbf{e} \cdot \mathbf{e}}{\dfrac{4}{t} \bar{J}_1\left(\dfrac{2}{t}\right)}. \tag{25}$$

A bound on W_v, the work done in volume change, can be found in a similar manner. The result obtained is given by

$$W_v = 3 \int_0^t p\dot{v}\, d\tau = \frac{3v^2}{\dfrac{4}{t} \bar{J}_2\left(\dfrac{2}{t}\right)}. \tag{26}$$

The total specific work done in deforming an isotropic linear viscoelastic material whose creep functions may be represented by equations (5) and (7) may therefore be bounded from below

$$W = W_s + W_v \geq \frac{t}{4} \left\{ \frac{\mathbf{e} \cdot \mathbf{e}}{\bar{J}_1\left(\dfrac{2}{t}\right)} + \frac{3v^2}{\bar{J}_2\left(\dfrac{2}{t}\right)} \right\}. \tag{27}$$

It is possible to derive a similar expression for a solid based on a generalized Maxwell model. In this case it is assumed that the relaxation functions $G_1(t)$, $G_2(t)$ for shear and volume change can be written in the form

$$G_1(t) = \sum_{j=1}^{n} E_j' e^{(-E_j'/\nu_j' t)}, \tag{28a}$$

and

$$G_2(t) = \sum_{j=1}^{n} K_j' e^{(-K_j'/\mu_j' t)}, \tag{28b}$$

where E_j', ν_j', K_j', μ_j' are non-negative constants. The total specific work is again bounded from below.

$$W \geq \frac{1}{t}\left(\mathbf{e}\cdot\mathbf{e}\bar{G}_1\left(\frac{2}{t}\right) + 3\,v^2\bar{G}_2\left(\frac{2}{t}\right)\right) \tag{29}$$

(27) and (29) are seen to be identical on observing that (e.g. Lee [4]) for any linear viscoelastic material

$$\bar{G}(p)\bar{J}(p) = \frac{1}{p^2}. \tag{30}$$

The work inequality given in (27) and (29) differs from any previous work of this kind known to the authors in that an attempt is made to use the time-independent elements of a model to develop a result for a time-dependent material. The final result is stronger than the inequalities used by Drucker and Breuer and Onat in that the work is bounded by a positive quantity rather than zero. In its present form it could be used to bound the minimum amount of work required to produce a given strain in a viscoelastic material in a given time. It must be emphasized, however, that the inequalities apply only to materials in which creep and relaxation functions can be expressed in the forms given in equations (5), (7) and (28). The number of terms in these expressions can be as large as required.

The restriction of isotropy is not essential in the development of the bound. Bounds can be obtained in a similar manner for anisotropic linear viscoelastic materials, provided that the creep and relaxation curves can be suitably expressed as an exponential series.

References

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