HEAVY ROTATING STRING IN A CASING*

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1. Introduction. It is well known in the linear theory that a heavy string with one endpoint free has eigenfunctions of lateral displacement only at certain eigenvelocities of rotation ω_n which form a discrete spectrum. Kolodner [1] has shown that according to the more accurate non-linear theory, a string can rotate at any velocity $\omega > \omega_1$, and that for each ω in the range $\omega_n < \omega \leq \omega_{n+1}$ there are exactly n distinct modes of rotational displacement. It has been found in [2] that even if the linear equation is used, a string can still rotate at any velocity $\omega > \omega_1$ if it is contained in a casing of small radius. Furthermore, for each ω in the range $\omega_n \leq \omega < \omega_{n+1}$ there are exactly n distinct modes of rotation. The essential feature of this phenomenon is that the string is subjected to a constraint condition giving rise to a moving boundary. This result will be demonstrated by the two theorems proved in this paper.

2. Formulation. Consider the rotation of a string of length L with its upper end ($x = L$) fixed and lower end ($x = 0$) free. If we let $y(x)$ be the steady state solution for lateral displacement, then

$$\frac{d}{dx}\left(\rho g\, x \frac{dy}{dx}\right) + \rho\omega^2 y = 0$$

is the linearized governing equation in which ρ is the density per unit length of the string and ω the angular velocity. Introducing $\xi = x/L$, $\eta = y/L$, $\Omega^2 = \omega^2 L/g$ we get

$$\frac{d}{d\xi}\left(\xi \frac{d\eta}{d\xi}\right) + \Omega^2\eta = 0. \tag{1}$$

This equation has a singularity at $\xi = 0$. Therefore, appropriate boundary conditions for the eigenvalue problem are

$$\eta(1) = 0, \qquad \eta(0) \text{ finite} \tag{2}$$

The eigenvelocities and mode shapes corresponding to equations (1) and (2) are Ω_n and $J_0(2\Omega_n\xi^{1/2})$ where $2\Omega_n$ is the nth zero of $J_0(Z)$.

If the string is constrained to rotate in a casing of dimensionless radius $r/L = \delta \ll 1$, then the boundary conditions (2) become

$$\eta(1) = 0, \qquad |\eta(\xi)| \leq \delta \ll 1 \qquad (0 \leq \xi \leq 1) \tag{3}$$

It will be shown that equations (1) and (3) have nontrivial solutions for any $\Omega > \Omega_1$. The existence of such solutions may easily be seen from the following auxiliary initial value problem.

We consider the function $u(\xi, a, \mu)$ on a real ξ interval, $I_\xi = (\xi;\ a \leq \xi \leq b < \infty)$, $0 < a < 1$, and a real μ interval $I_\mu = (\mu;\ \Omega_1^2 \leq \mu \leq M < \infty)$ such that

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$$(\xi, u(\xi, a, \mu), u'(\xi, a, \mu)) \varepsilon D, \left(\xi \varepsilon I_\xi, ' = \frac{\partial}{\partial \xi}\right),$$

$$(\xi u')' + \mu u = 0, \tag{4}$$

$$u(a, a, \mu) = \delta, \tag{5}$$

$$u'(a, a, \mu) = 0. \tag{6}$$

We now state the following theorem for the above defined initial value problem:

Theorem 1. (1) $u(\xi, a, \mu)$ exists, is unique, and depends continuously on a and μ.

(2) $u(\xi, a, \mu)$ has an infinite number of isolated zeros $Z_n(a, \mu)$, $a < Z_1 < Z_2 < \cdots < Z_n$ and $\lim_{n\to\infty} Z_n(a, \mu) = \infty$; furthermore, $Z_n(a, \mu)$ is a monotonically decreasing function of μ for constant a.

(3) If we define $\mu_n(a)$ to be such that $Z_n(a, \mu_n(a)) = 1$, then there are an infinite number of μ_n, $\mu_1(a) < \mu_2(a) < \cdots < \mu_n(a)$ and $\lim_{a\to 0} \mu_n(a) = \Omega_n^2$, $\lim_{a\to 1} \mu_n(a) = \infty$.

(4) $u_n(\xi, a) = u(\xi, a, \mu_n(a))$ has exactly n zeros in the interval $[a, 1]$ and $\max |u_n(\xi, a)| = u_n(a, a) = \delta$.

(5) The $\mu_n(a)$ are differentiable functions of a and $d\mu_n/da > 0$; furthermore, $\lim_{a\to 1} d\mu_n/da = \infty$.

Proof: (1) Existence and uniqueness of $u(\xi, a, \mu)$.

Rewrite equation (4) as

$$u'' = -\frac{1}{\xi}(u' + \mu u) = f(\xi, u, u', \mu). \tag{7}$$

Let D_μ be the domain of (ξ, u, u', μ) space

$$D_\mu : (\xi, u, u') \varepsilon D \qquad \mu \varepsilon I_\mu ;$$

then f is continuous on D_μ and

$$|f(\xi, u, u', \mu) - f(\xi, u_1, u_1', \mu)|$$
$$\leq \frac{\mu}{\xi}(|u - u_1| + |u' - u_1'|) \leq \frac{M}{a}(|u - u_1| + |u' - u_1'|)$$

since $\mu \geq \Omega_1^2 > 1$. Thus f satisfies a Lipschitz condition in u and u' uniformly with respect to μ on D_μ and Part (1) of Theorem 1 follows from [3].

(2) Zeros of $u(\xi, a, \mu)$:

A zero of a nontrivial solution of equation (4) is isolated. Indeed, let the solution u vanish at ξ_0. Then $u'(\xi_0) \neq 0$, for otherwise u'' and hence $u^{(k)}$ vanish at ξ_0 for all k and $u \equiv 0$. This proves that ξ_0 is an isolated zero.

The unique solution of equations (4, 5, 6) can be expressed explicitly in terms of Bessel functions

$$u(\xi, a, \mu) = [J_1(2\mu^{1/2}a^{1/2})Y_0(2\mu^{1/2}\xi^{1/2}) - Y_1(2\mu^{1/2}a^{1/2})J_0(2\mu^{1/2}\xi^{1/2})]\mu^{1/2}a^{1/2}\pi\delta. \tag{8}$$

Since the Bessel functions have an infinite number of zeros, it is clear that u has an infinite number of zeros at $\xi = Z_n(a, \mu)$ and $\lim_{n\to\infty} Z_n(a, \mu) = \infty$.

(3) $Z_n(a, \mu)$ decreases monotonically with μ for constant a.

If we let

$$\Delta(\xi, a, \mu) = J_1(2\mu^{1/2}a^{1/2})Y_0(2\mu^{1/2}\xi^{1/2}) - Y_1(2\mu^{1/2}a^{1/2})J_0(2\mu^{1/2}\xi^{1/2})$$

then, by definition of Z_n and equation (11), $\Delta_a(Z_n, \mu) = \Delta(Z_n, a, \mu) = 0$. Since $u(\xi, a, \mu)$ depends continuously on its arguments so does $\Delta_a(Z_n, \mu)$. We may now formally differentiate $\Delta_a(Z_n, \mu)$ to get

$$\left(\frac{\partial Z_n}{\partial \mu}\right)_a = -\frac{\partial \Delta_a}{\partial \mu} \Big/ \frac{\partial \Delta_a}{\partial Z_n} = \frac{Z_n}{\mu}\left\{\left[\frac{J_0(2\mu^{1/2}Z_n^{1/2})}{J_1(2\mu^{1/2}a^{1/2})}\right]^2 - 1\right\}.$$

For μ large enough, we may approximate Bessel functions by the first term of their asymptotic expansions and the zeros of equation (8) are approximately

$$2\mu^{1/2}Z_n^{1/2} \simeq 2\mu^{1/2}a^{1/2} + \frac{2n-1}{2}\pi;$$

thus,

$$\left|\frac{J_0(2\mu^{1/2}z_n^{1/2})}{J_1(2\mu^{1/2}a^{1/2})}\right| \simeq \left(\frac{a}{Z_n}\right)^{1/4} < 1$$

since $Z_n > a$. It follows that $(\partial Z_n/\partial\mu)_a < 0$ for μ large. However, since Z_n are the roots of $\Delta(\xi, a, \mu) = 0$, $(\partial Z_n/\partial\mu)_a$ can never vanish unless $\mu = \infty$. Thus, $(\partial Z_n/\partial\mu)_a < 0$ must hold for all values of μ. This proves that $Z_n(a, \mu)$ is a monotonically decreasing function of μ for constant a.

(4) $\mu_n(a)$ and $\lim_{a\to 0} \mu_n(a) = \Omega_n^2$, $\lim_{a\to 1} \mu_n(a) = \infty$:

Since there are an infinite number of $Z_n(a, \mu)$, it is clear that an infinite number of μ_n with $Z_n(a, \mu_n(a)) = 1$ (or $\Delta(1, a, \mu_n(a)) = 0$) exist. Furthermore, since $Z_1(a, \mu) < Z_2(a, \mu) < \cdots < Z_n(a, \mu)$ and $(\partial Z_n/\partial\mu)_a < 0$, the $\mu_n(a)$ must have the relation $\mu_1(a) < \mu_2(a) < \cdots < \mu_n(a)$

To determine the limits of $\mu_n(a)$ as a approaches zero, we note that $\mu_n(a)$ satisfies $\Delta(1, a, \mu_n(a)) = 0$. Therefore, if $\mu_n^0 = \lim_{a\to 0} \mu_n(a)$, we must have

$$\lim_{a\to 0} \Delta(1, a, \mu_n(a)) = J_0(2\mu_n^{0\,1/2}) = 0;$$

thus,

$$\lim_{a\to 0} \mu_n(a) = \mu_n^0 = \Omega_n^2 .$$

Similarly, if $\mu_n^1 = \lim_{a\to 1} \mu_n(a)$, we must have

$$\lim_{a\to 1} \Delta(1, a, \mu_n(a)) = -1/\pi(\mu_n^1)^{1/2} = 0;$$

thus,

$$\lim_{a\to 1} \mu_n(a) = \mu_n^1 = \infty .$$

(5) $u_n(\xi, a) = u(\xi, a, \mu_n(a))$ has n zeros in $[a, 1]$ and $\max |u_n(\xi, a)| = \delta$.

By definition, $\mu_n(a)$ satisfies $Z_n(a, \mu_n(a)) = 1$. It is clear that $u_n(\xi, a)$ has exactly n zeros in the interval $[a, 1]$. These zeros are $Z_{ni}(a) = Z_i(a, \mu_n(a))$ where $i = 1, 2, 3, \cdots, n$. Explicitly:

$$u_n(\xi, a) = [J_1(2\mu_n^{1/2}a^{1/2})Y_0(2\mu_n^{1/2}\xi^{1/2}) - Y_1(2\mu_n^{1/2}a^{1/2})J_0(2\mu_n^{1/2}\xi^{1/2})]\mu_n^{1/2}a^{1/2}\pi\delta.$$

This solution holds only for $a > 0$. When $a = 0$ the solution is $\delta J_0(2\Omega_n\xi^{1/2})$.

Comparing $u_n(\xi, a)$ with the equality

$$Y_0(z)J_1(z) - J_0(z)Y_1(z) = 2/\pi z,$$

we find that $u_n(\xi, a) = \delta$ if and only if $\xi = a$. It follows that $u_n(\xi, a) < \delta$ for $a < \xi < 1$ since $u_n(1, a) = 0$.

(6) $\mu_n(a)$ are differentiable functions of a and $d\mu_n/da > 0$.

By definition of $\mu_n(a)$, we have $\Delta_1(a, \mu_n(a)) = \Delta(1, a, \mu_n(a)) = 0$. Formally differentiating $\Delta_1(a, \mu_n(a))$, we get

$$\frac{d\mu_n}{da} = -\frac{\partial\Delta_1}{\partial a}\bigg/\frac{\partial\Delta_1}{\partial\mu_n} = \frac{\mu_n}{a}\bigg/\left\{\left[\frac{J_1(2\mu_n^{1/2}a^{1/2})}{J_0(2\mu_n^{1/2})}\right]^2 - 1\right\}.$$

Since μ_n and a must satisfy the equation $\Delta(1, a, \mu_n(a)) = 0$, the denominator of the last equation never vanishes and hence $d\mu_n/da$ exists. To prove $d\mu_n/da > 0$, we use the same argument as in the proof of part (2). For μ_n large enough, equation $\Delta(1, a, \mu_n(a)) = 0$ may be approximated by the first term of its asymptotic expansion to give

$$2\mu_n^{1/2} \simeq 2\mu_n^{1/2}a^{1/2} + \frac{2n-1}{2}\pi$$

thus,

$$\left|\frac{J_1(2\mu_n^{1/2}a^{1/2})}{J_0(2\mu_n^{1/2})}\right| \simeq \left(\frac{1}{a}\right)^{1/4} > 1$$

since $a < 1$. It follows that $d\mu_n/da > 0$ for μ_n large. However, $d\mu_n/da$ can never vanish since $\mu_n \geq \Omega_1^2 \neq 0$. Thus $d\mu_n/da > 0$ for all a. Furthermore, $\lim_{a\to 1} d\mu_n/da = \infty$ since $\lim_{a\to 1} \mu_n(a) = \infty$ from part (3). We have thus completed the proof of part (5). The result of part (5), *Theorem 1,* is shown graphically in Fig. 1.

Theorem 2. The function $\eta(\xi)$ on the interval (0, 1) satisfying

$$\left.\begin{aligned} &(\xi\eta')' + \Omega^2\eta = 0 \\ &\eta(1) = 0, \qquad |\eta(\xi)| \leq \delta \end{aligned}\right\} \xi \in I'_\xi(\xi;\, \eta(\xi) \not\equiv \delta) \tag{9}$$

$$\eta(\xi) = \delta \qquad\qquad \xi \notin I'_\xi \tag{10}$$

has for any $\Omega_n \leq \Omega < \Omega_{n+1}$ exactly n nontrivial solutions $\eta_1, \eta_2, \cdots, \eta_n$ such that

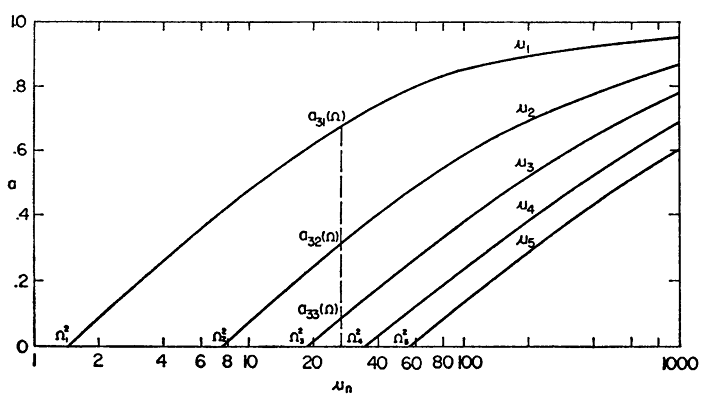

FIG. 1.

$$\eta_i(\xi) = \begin{cases} \delta & 0 \le \xi \le a_{ni} \\ u_i(\xi, a_{ni}) & a_{ni} \le \xi \le 1 \end{cases} \qquad (i = 1, 2, \cdots, n)$$

where $a_{ni}(\Omega)$ are determined by $\Omega^2 = \mu_i(a_{ni}(\Omega))$. The solution $\eta_i(\xi)$ has exactly i isolated zeros.

Proof: From *Theorem 1*, the $\mu_n(a)$ are monotonically increasing functions of a with the properties that $\mu_1(a) < \mu_2(a) < \cdots$ and $\lim_{a\to 0} \mu_n(a) = \Omega_n^2$. Thus for $\Omega_n \le \Omega < \Omega_{n+1}$, there are n values of a, $a_{n1}(\Omega) > a_{n2}(\Omega) > \cdots > a_{nn}(\Omega) \ge 0$, such that $\Omega^2 = \mu_i(a_{ni}(\Omega))$, $i = 1, 2, \cdots, n$. Corresponding to each a_{ni} there is a $u_i(\xi, a_{ni})$ defined in $a_{ni} \le \xi \le 1$ satisfying equation (9) and the boundary conditions $u_i(a_{ni}, a_{ni}) = \delta$ and $u_i(1, a_{ni}) = 0$; furthermore, $u_i'(a_{ni}, a_{ni}) = 0$ and $\max |u_i(\xi, a_{ni})| = \delta$. Therefore

$$\eta_i(\xi) = \begin{cases} \delta & 0 \le \xi \le a_{ni} \\ u_i(\xi, a_{ni}) & a_{ni} \le \xi \le 1 \end{cases} \qquad (i = 1, 2, \cdots, n)$$

are the solutions of equations (9) and (10). Since $u_i(\xi, a_{ni})$ has exactly i isolated zeros, $\eta_i(\xi)$ also has i isolated zeros.

Let us take for example $\Omega_3 < \Omega < \Omega_4$, Fig. 1. There are three possible values for a, i.e., $a_{31}(\Omega)$, $a_{32}(\Omega)$ and $a_{33}(\Omega)$. The three possible modes, one corresponding to each value of a, are

$$\eta_i(\xi) = \begin{cases} \delta & 0 \le \xi \le a_{3i} \\ u_i(\xi, a_{3i}) & a_{3i} \le \xi \le 1 \end{cases} \qquad (i = 1, 2, 3).$$

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References

1. I. I. Kolodner, *Heavy Rotating String—A Nonlinear Eigenvalue Problem*, Communications on Pure and Applied Mathematics, **8**, 395–408, 1955
2. C. H. Wu, *Critical Speeds of a Rotating Drill Shaft in a Uniform Casting*, Technical Report ACS-PRF Grant #1376-A2, University of Minnesota, 1964
3. E. A. Coddington, and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955 Section 7, p. 22