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SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. Marginal instructions to the printer should be written in pencil to distinguish them clearly from the body of the text.

The papers should be submitted in final form. Only typographical errors may be corrected in proofs; composition charges for all major deviations from the manuscript will be passed on to the author.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical Work: As far as possible, formulas should be typewritten; Greek letters and other symbols not available on the typewriter should be carefully inserted in ink. Manuscripts containing pencilled material other than marginal instructions to the printer will not be accepted.

The difference between capital and lower-case letters should be clearly shown; care should be taken to avoid confusion between zero (0) and the letter O, between the numeral one (1), the letter l and the prime ($'$), between alpha and a, kappa and k, mu and u, nu and v, eta and n.

The level of subscripts, exponents, subscripts to subscripts and exponents in exponents should be clearly indicated.

Dots, bars, and other markings to be set *above* letters should be strictly avoided because they require costly hand-composition; in their stead markings (such as primes or indices) which *follow* the letter should be used.

Square roots should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{\ }$.

Complicated exponents and subscripts should be avoided. Any complicated expression that recurs frequently should be represented by a special symbol.

For exponentials with lengthy or complicated exponents the symbol exp should be used, particularly if such exponentials appear in the body of the text. Thus,

$$\exp [(a^2 + b^2)^{1/2}] \text{ is preferable to } e^{(a^2+b^2)^{1/2}}$$

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus,

$$\frac{\cos (\pi x/2b)}{\cos (\pi a/2b)} \text{ is preferable to } \frac{\cos \frac{\pi x}{2b}}{\cos \frac{\pi a}{2b}}$$

In many instances the use of negative exponents permits saving of space. Thus,

$$\int u^{-1} \sin u \, du \text{ is preferable to } \int \frac{\sin u}{u} \, du.$$

Whereas the intended grouping of symbols in handwritten formulas can be made clear by slight variations in spacing, this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus,

$$(a + bx) \cos t \text{ is preferable to } \cos t \, (a + bx).$$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of parentheses, brackets and braces. Thus,

$$\{[a + (b + cx)^n] \cos ky\}^2 \text{ is preferable to } ((a + (b + cx)^n) \cos ky)^2.$$

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

Bibliography: References should be grouped together in a Bibliography at the end of the manuscript. References to the Bibliography should be made by numerals between square brackets.

The following examples show the desired arrangements: (*for books*—S. Timoshenko, *Strength of materials*, vol. 2, Macmillan and Co., London, 1931, p. 237; *for periodicals*—Lord Rayleigh, *On the flow of viscous liquids, especially in three dimensions*, Phil. Mag. (5) 36, 354–372 (1893). Note that the number of the series is not separated by commas from the name of the periodical or the number of the volume.

Authors' initials should precede their names rather than follow it.

In quoted titles of books or papers, capital letters should be used only where the language requires this. Thus, *On the flow of viscous fluids* is preferable to *On the Flow of Viscous Fluids*, but the corresponding German title would have to be rendered as *Über die Strömung zäher Flüssigkeiten*.

Titles of books or papers should be quoted in the original language (with an English translation added in parentheses, if this seems desirable), but only English abbreviations should be used for bibliographical details like ed., vol., no., chap., p.

Footnotes: As far as possible, footnotes should be avoided. Footnotes containing mathematical formulas are not acceptable.

Abbreviations: Much space can be saved by the use of standard abbreviations like Eq., Eqs., Fig., Sec., Art., etc. These should be used, however, only if they are followed by a reference number. Thus, "Eq. (25)" is acceptable, but not "the preceding Eq." Moreover, if any one of these terms occurs as the first word of a sentence, it should be spelled out.

Special abbreviations should be avoided. Thus "boundary conditions" should always be spelled out and not be abbreviated as "b.c.," even if this special abbreviation is defined somewhere in the text.

CONTENTS

BOOK REVIEWS

Stress analysis: recent developments in numerical and experimental methods. Edited by O. C. Zienkiewicz and G. S. Holister. John Wiley & Sons, Ltd., London and New York, 1965. X + 469 pp. $15.50.

Part I of the timely volume contains nine chapters on numerical methods. The first two of these are respectively devoted to the finite difference approach in general (D. N. de G. Allen and D. W. Windle) and to its implementation in two-dimensional elasticity and plate theory (O. C. Zienkiewicz). The next four chapters contain a careful derivation and discussion of the fundamental equations of shell theory (D. N. de G. Allen and G. M. Birtwistle) without reference to the numerical treatment of these equations. Chapters 7 and 8 are concerned with finite element procedures in structural mechanics (R. W. Clough) and in the solution of plate and shell problems (O. C. Zienkiewicz). Chapter 9 is a masterly exposition of the complete duality between displacements and equilibrium models in the finite element method (B. Fraeijs de Veubeke), and Chapter 10 is devoted to the use of integral equations in stress analysis (C. E. Massonnet).

Part II, which is perhaps of lesser interest to the majority of readers of this Quarterly, contains seven chapters on experimental methods: Some new techniques in strain measurement (V. M. Hickson); Photoelastic techniques applied to rock mechanics problems of underground excavations and foundations (I. Hawkes and G. S. Holister); Three-dimensional elasticity (C. Snell); Grid and moiré methods of stress analysis (J. P. Duncan); Electrical analogue methods of structural analysis (S. C. Redshaw); Structural model techniques (M. Rocha); and In situ strain and stress measurements (M. Rocha).

William Prager

Introduction to numerical analysis. By Carl-Erik Fröberg. Addison-Wesley Publishing Co., Reading, Massachusetts, 1965. X + 340 pp. $8.95.

This translation from the Swedish will doubtless be widely adopted as text for a first course in numerical analysis. The required background consists of elementary calculus and differential equations. Although there are relatively few explicit references to automatic computation, the choice of the methods discussed in the book has been strongly influenced by their suitability for automatic computers. The exercises, however, require only a desk calculator and function tables.

The following chapter headings (with abbreviated selected section headings) will give an idea of the scope of the book. Numerical Computations (representation of numbers, on errors, computation of functions, asymptotic series), Equations (cubic and quartic equations, Horner's scheme, Newton-Raphson, Bairstow, and Graeffe methods, iterative methods, Q. -D. method), Matrices (matrix operations, eigenvalues and eigenvectors, matrix functions, norms), Linear Systems of Equations (methods of Gauss and Crout, error estimates, iterative methods, conjugate gradient method), Matrix Inversion (Jordan and Cholesky, escalator method, iterative methods), Algebraic Eigenvalue Problems (Jacobi, Lanczos, Givens, and Householder methods), Linear Operators, Interpolation (Lagrange and Hermite formulas, difference schemes, subtabulation, inverse interpolation), Numerical Differentiation, Numerical Quadrature (Cotes, Gauss, and Chebyshev formulas), Summation (Bernoulli numbers and polynomials, Euler transformation), Multiple Integrals, Difference Equations, Ordinary Differential Equations (Runge-Kutta, Milne, and Numerov methods, boundary and eigenvalue problems), Partial Differential Equations (hyperbolic, parabolic, and elliptic equations, eigenvalue problems, stability), Approximation (least square polynomial approximation, approximation with orthogonal polynomials, trigonometric, exponential functions, with Chebyshev polynomials, continued fractions, and rational functions), the Monte Carlo Method (random numbers, random walks, computation of definite integrals, simulation), Linear Programming (simplex method, transportation problem, quadratic, convex, and dynamic programming). As this list indicates, the book provides a comprehensive tour of modern numerical analysis.

W. Prager

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BOOK REVIEWS

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Absolute stability of regulator systems. By M. A. Aizerman and F. R. Gantmacher. Holden-Day, Inc., San Francisco, London, Amsterdam, 1964. 172 pp. $8.95.

During the last twenty years a system of nonlinear ordinary differential equations known as the Lurie equations has drawn the attention of many mathematicians and engineers. The equations arise from practical considerations in control theory and from the fundamental mathematical question: to what extent does linearization predict stability for a nonlinear system? Recently a great deal of progress has been made concerning the question of asymptotic stability in the large for this system. This book surveys the history and gives a thorough treatment of the main results.

It is fitting that Professor Aizerman should be coauthor in such a work since he has been closely connected with the problem since its first conception. Indeed this is reflected in the book by a rich store of details and an extensive bibliography (which was brought up to date — 1964 — by the translator). Approximately the first third of the book is concerned with the early work of Lurie, Letov, Malkin, Lefschetz et al. on the construction of Liapunov functions by algebraic methods. Here one finds a very careful survey of the highlights of the early work.

The middle of the book treats the work of V. M. Popov and his frequency domain criterion for asympototic stability. It was the work of Popov in the late fifties and early sixties that changed the nature of the machinery and bred new life into the problem. In this section one finds the core and the value of this book. Popov's method is used to yield stability criteria for the non-critical case and by a somewhat artificial trick even the critical case. Careful instructions are given on graphical methods for using the frequency domain criteria.

The end of the book considers the connection between the Liapunov method and the Popov method. It is unfortunate that much of the fundamental machinery necessary for understanding this aspect of the problem was still under development at the time of writing. Even the period between submission of the manuscript and the receipt was fruitful in this respect. The authors found it necessary to add in proof an appendix and many footnotes to explain that a particular question raised in the body had been resolved by the works of Kalman and Yacubovich — then just published. Although large parts of this section were obsolete even at the time of the first Russian printing the book is so carefully written and full of detail that the discerning reader can find much to interest him.

One cannot help from remarking that the otherwise pleasant style of the writing is marred by the lack of matrix notation.

Kenneth Meyer

Statistical theory and methodology, in science and engineering. By K. A. Brownlee. Second Edition. John Wiley & Sons, Inc., New York, London, Sydney, 1965. xvi + 590 pp. $18.50.

The second edition of a Wiley Publication In Statistics (of a known and well used first edition) is indeed a very impressive book written with the dexterity of a master. It touches on most important subjects in statistics and is highly informative. With its many exercises and answers, it should serve as a textbook for upper undergraduate classes. It has 17 chapters and an appendix of 11 tables. Starting with a well motivated chapter called Mathematical Ideas, the author points out that the book is principally concerned with statistics and not with probability. However, the latter topic occupies the first 86 pages of the book in a good account. Chapter 2 which is called Statistical Ideas gives the basic concepts of statistics which the author treats with substantial smoothness and continuity. Then the book moves on to a discussion of the Binomial, Hypergeometric, and Poisson Distributions leading naturally to an introduction to queues in equilibrium. Multinomial Distributions, Testing Randomness Hypotheses, and Nonparametric tests occupy three chapters. Several chapters are concerned with regression and with the analysis of variance which are essential for application purposes. A chapter on Partially Hierarchial Situations and another on Simple Experimential Designs are also included. One cannot fail to be impressed by this useful text and the facility which the author shows in conveying so many ideas so well. It is recommended both as a text and for reference purposes.

T. L. Saaty

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BOOK REVIEWS

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La programmation dynamique: Gestion scientifique séquentielle. By A. Kaufmann and R. Cruon. Preface by H. P. Galliher. Dunod, Paris, 1965. XVI + 275 pp. $11.87.

The book is concerned with discrete dynamic programming. Chapters 1 and 2 respectively deal with deterministic programs over finite and infinite horizons. The corresponding material for stochastic programs is presented in Chapter 3 and 4. Chapter 5 is devoted to dynamic programs in form of finite Markov chains, and Chapter 6 briefly discusses the generalizations of these models implied by non-sequential structures and nonadditive values. The book, which appears to be the first French text in this field, is primarily meant for engineers and economists without prior knowledge of the subject. The exposition is particularly clear, and the general theory is amply illustrated by numerical examples, some of which are worked out in great detail.

W. Prager

Measure and integration. By Sterling K. Berberian. The Macmillan Co., New York, and Collier-Macmillan Ltd., London, 1965. xviii + 312 pp. $9.95.

Contents: I Measures, II Measurable Functions, III Sequences of Measurable Functions, IV Integral Functions, V Convergence Theorems, VI Product Measures, VII Finite Signed Measures, VIII Integration Over Locally Compact Spaces, IX Integration Over Locally Compact Groups.

This text covers the standard material in Halmos' "Measure Theory" and has little to recommend it over that text. Ample exercises are provided and interesting and useful notes are appended. Some theorems have lost their standard historical names in this book.

R. S. Bucy

Operations research: process and strategy. By David Stoller. University of California Press, Berkeley, Los Angeles, 1964. vii + 159 pp. $5.00.

A considerable need for educating people in the methods of Operations Research is fulfilled in this well written book. Its 23 chapters are divided into three parts, the first part called Characteristics of Operations Research includes three chapters. The second part called Servicing has ten chapters on queues and on maintenance. The final part on Strategy is concerned with game theory and linear programming. Other optimization techniques of importance in the field are not included perhaps because of the choice of Strategy as the underlying motivation of the last part.

The development of the topics is uniquely lucid. The chapters are short, but convey the ideas adequately. The book is punctuated with illustrative examples. It should be an excellent text for a short course on the subject to individuals whose training does not permit detailed explorations. It is well motivated and is recommended reading.

T. L. Saaty

Boolean algebras. By Roman Sikorski. Second Edition. Academic Press Inc., New York, and Springer-Verlag, Berlin, Göttingen, Heidelberg, 1964. x + 237 pp. $9.50.

This scholarly monograph has become a standard reference and should be present in the library of every mathematician whose work touches upon Boolean Algebra. The intention of the author was to give a survey of our present knowledge of the subject. The first chapter deals with Boolean Algebras with finitary operations only; infinitary operations are treated in the second chapter. All details of the definitions and proofs are present. The set-theoretical aspects are stressed, somewhat to the expense of the algebraic ones. Examples from, and applications to, set theory, topology, probability, measure theory, logic, etc. abound and are either integrated into the text or sketched in one of the nine appendices. The nineteen page updated bibliography should prove of particular value.

E. Engeler

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BOOK REVIEWS

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The problem of the minimum of a quadratic functional. By S. G. Mikhlin. Translated by A. Feinstein. Holden-Day, Inc., San Francisco, London, Amsterdam, 1965. ix + 155 pp. $8.95.

The setting for this monograph, which is a translation of the Russian edition first appearing in 1952, is a complex Hilbert space $\mathcal{H}$ and a densely defined positive ($(Au, u) > 0$ for $u \neq 0$) linear operator $A : \mathcal{H} \to \mathcal{H}$. For given $f \in \mathcal{H}$, an element $u \in \mathcal{H}$ is a solution of the problem $Au = f$ if and only if it provides a minimum value to the quadratic functional $F(u) = (Au, u) - (u, f) - (f, u)$. This should clarify the precise meaning of the book's title; the applications consist mainly of the case where A is an elliptic partial differential operator.

From the start, the text assumes a reasonable knowledge of functional analysis (a welcome change from the large number of books beginning with a one chapter summary of functional analysis, enough to confuse the uninitiated and bore the remaining readers) for example on page 3 it is stated "Obviously, every positive operator is symmetric" a result true in the present setting but not true in a real Hilbert space.

A major part of the effort concerns the problem of extending a positive (or positive definite $(Au, u) \geq k(u, u)$) operator A to a self adjoint operator, then proving existence, and obtaining methods for constructing solutions, of $Au = f$, where A again denotes the self adjoint extension of the original operator. The field of extending closed operator, and associated notions of weak solutions, received much attention in the middle to late 1950's. Since the original manuscript was completed in 1951, it predates the work of Hörmander, F. E. Browder, etc., and these advances are neither mentioned (or could be done justice) in a five page supplement included to update the original text. Nevertheless the exposition, notation and clarity with which problems and methods are presented makes this an excellent book for learning the basic methods for extending the operators of mathematical physics.

Chapter one begins with the Friedrichs extension of positive definite operators to self adjoint operators; next considers positive operators, and illustrates by example the differences in the two cases. It continues with the method of least squares, variational methods in eigenvalue problems and Galerkin's method.

Chapter two deals with auxiliary results to be used in later chapters. Again, the major ideas and main consequences are presented with the utmost clarity. This chapter deals with the averaging operator (Friedrich's mollifier), the generalized derivative of Sobolev, properties of singular integrals, Sobolev's integral identity which gives an expression for a function defined in a star shaped region in terms of its nth derivatives and a polynomial of degree less than or equal to $n - 1$, and several other short topics.

Chapter three deals with elliptic differential operators, applying the results of chapters one and two. A short section on regularity conditions is included.

The final chapter concerns applications to elasticity theory.

In summary, the book is *readable,* stressing major ideas rather than worrying about the most general setting. For the reader interested in the applications of functional analysis to partial differential equations, it is, in this reviewer's opinion, the best starting point in the literature to date.

H. Hermes

Elements of ordinary differential equations. By Michael Golomb and Merrill Shanks. Second Edition. McGraw-Hill Book Co., New York, St. Louis, San Francisco, Toronto, London, 1965. xi + 410 pp. $8.95.

This is an excellent book to use for an introductory course in differential equations of either one semester or two semester duration, depending upon the caliber of the students. The exposition is careful and leisurely with many examples in the text and in the exercises. There is considerable emphasis upon the physical origin of various classes of differential equations, upon numerical solution, and upon geometric concepts. The book is heartily recommended.

Richard Bellman

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BOOK REVIEWS

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The mathematics of matrices. By Philip J. Davis. Blaisdell Publishing Co., New York, Toronto, London, 1965. xiii + 348 pp. $7.50.

The author has undertaken to develop a course in matrix theory and linear algebra for high school seniors and for college freshmen, and his approach to the subject is, he says, "with the eye of the analyst and the experience of the applied mathematician." He is certainly qualified in both respects.

The style is quite informal and often conversational ("Felix Klein, the Grand Mogul of Mathematics at the beginning of the twentieth century"); the figures (e.g., "A camel, sheared") and suggested applications (e.g. "The analysis of nuts") frequently wryly humorous; there are numerous historical notes. General theorems are stated but few matrices are exhibited of order greater than three.

The first six chapters are "the core of the course." These describe the standard operations, with determinants and inner products. Chapter 7, "Matrices as operators," is "for flavor and philosophy." Chapter 8 gives the Cayley-Hamilton theorem, but not normal forms. Chapter 9 deals briefly with abstract algebra. And Chapter 10, "Pippins and cheese" contains supplementary problems and "topics for investigation," touching on spin matrices, normal matrices, quaternions, magic squares, and other items.

To this reviewer, the most serious omission seems to be conics. Since algebra, trigonometry, and plane analytic geometry are presupposed (though "in modest amounts"), it would seem that their introduction in Chapter 8 would have been well justified, even if at the expense of some of the other applications introduced here and earlier, and would have permitted a more extensive development of the notion of characteristic roots.

Nevertheless, one can only applaud any effort to make matrices accessible at an earlier stage in the educational process, and this introduction should succeed very well.

A. S. Householder

Math and aftermath. By Robert Hooke and Douglas Shaffer. Walker and Company, New York, 1965. x + 233 pp. $5.95.

This is the fifth volume in the Westinghouse "Search Books" series and was prepared under the supervision of the editorial board of the Westinghouse Research Laboratories.

We have here a collection of informal essays, directed toward a vaguely defined general audience, and with the goal of telling what applied mathematics is about. Thus, among other things, there are discussions of Kepler's Laws and gravitation, statistics, vibration phenomena, linear programming, probability, and the nature of mathematical models. There is no attempt to teach anything formally, but every so often a derivation has been put in. In numerous places the mathematics is far in advance of the probable audience. This is legitimate, for in expository writing, total comprehension often leads to total indifference. Besides, the reading public likes to see the little squiggles that contain the keys to the universe.

The essays are augmented by photographs and drawings, and an appendix containing a brief review of calculus, short biographies of several dozen scientists, and a substantial list of books for further reading. There is a beautiful hooked rug on the jacket representing the newly discovered 10 × 10 Greco-Latin Square.

The essays are done with a light touch; and if the authors have produced a one-shot "read and run" book instead of a many-shot "think" book, this can be attributed to the mission of the series.

An excellent volume to place in the hands of an undergraduate who is wavering between a career in mathematics or one in law, or in law hands of a lawyer who hankers after math and needs to be convinced that his choice was also correct.

Philip J. Davis