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GENERALIZED AERODYNAMIC FORCES ON AN OSCILLATING CYLINDRICAL SHELL*

By

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Nomenclature

a —Speed of sound
B —$(M^2 - 1)^{1/2}$
G_{mr} —See Eq. (17)
k —$\omega L/U$, reduced frequency
L —Cylinder length
M —Mach number
m —Streamwise mode number
n —Circumferential mode number
p —Aerodynamic Pressure
P —$(p/\cos n\theta)e^{-i\omega t}$, pressure amplitude
P_m —Pressure amplitude of mth mode
Q_{mr} —$\int_0^1 P_m(x)\psi_r(x)\, dx/\rho U^2$
R —Radius of cylinder
r —Radial coordinate; also streamwise mode number
s —Laplace transform variable
t —Time
U —Free-stream velocity
W —Radial deflection amplitude
w —$W(x) \cos n\theta\, e^{i\omega t}$, radial deflection
x —Streamwise coordinate
ζ —$(R/L)(M^2(s + ik)^2 - s^2)^{1/2}$
θ —Angular polar coordinate
ρ —Density
Φ —$(\varphi/\cos n\theta)e^{-i\omega t}$, velocity potential amplitude
φ —Velocity potential
$\psi_m(x)$—Streamwise mode shape
ω —Frequency

Superscripts

$*$ —Laplace transform
$'$ —differentiation

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Abstract. The present paper presents a mathematical and numerical solution for the problem of determining the aerodynamic forces on a harmonically oscillating cylindrical shell at supersonic speeds within the framework of the classical, linearized, potential flow theory. The method of solution is given in detail and extensive numerical results are presented to indicate the nature of the aerodynamic forces. Comparisons of the present results are made with those of simpler, but more approximate theories, such as the quasi-steady, two-dimensional theory and a (generalized) "slender-body" theory, to indicate where these may be used with confidence.

1. Introduction. The determination of the aerodynamic forces on an oscillating cylindrical shell of finite length due to an external flow is a prerequisite for the study of its aeroelastic stability. (See Figure 1.) Such a determination is the subject of the present paper. This problem has been previously considered by Holt and Strack [1] and, also, in a more recent report by Stearman [2]. The major assumptions in the mathematical model are that the flow is inviscid, irrotational (i.e., a velocity potential exists), supersonic, and that linearization is permissible. In both references [1] and [2] a formal mathematical solution is obtained in terms of the Laplace transform of the velocity potential. The use of the Laplace transformation (in the streamwise coordinate) had been used previously by Randall [3] in his investigation of the steady flow about quasi-cylindrical geometries and references [1] and [2] make use of his results in their investigation of the unsteady flow problem. The unsteady problem is inherently more difficult than the steady one, of course, and thus the authors of references [1] and [2] have made further approximations in an effort to obtain useful results. The approximation used by Holt and Strack is rather severe and essentially consists of, first, the reduction to steady flow from unsteady flow, and, secondly, an expansion about the limiting case of two-dimensional flow (i.e., an expansion in terms of the length-to-radius ratio of the cylinder). Only the first two terms in the expansion series are retained, the first of which is the well-known Ackeret result and the second the first-order correction to that result. In a more recent paper, Dżygadlo [4] has carried out a similar process in a more systematic fashion retaining the first three terms in the expansion and *including* the unsteady effects. He has noted that this expansion process is not rapidly convergent for typical flow and cylinder parameters. Most recently, Stearman has identified the inverse Laplace transform of the velocity potential "influence function" of the unsteady problem in terms of that of the steady problem, which has been computed and tabulated by Randall. In principle, all that is required then is a sufficiently accurate tabulation of the influence function for steady flow and an integration to compute aerodynamic pressure or two integrations to compute aerodynamic generalized forces. However, these integrations are rather awkward and it turns out that one needs the derivative of the steady flow influence function for the unsteady flow problem as well. For these reasons Stearman has suggested an alternative method whereby the Laplace transform of the influence function is approximated by a simpler function whose inverse can be determined analytically and which permits the integrations to be performed more simply. The approximation of the Laplace transform is made in a manner suggested by Luke [5]. The method would seem feasible although a considerable amount of numerical work is still required and the method does require a further approximation beyond the original problem formulation. To date, no results for the aerodynamic forces have been published. Finally, it is noted that the similar problem of a supersonic flow *inside* a cylindrical shell has been studied in reference [6] and a formal solution effected by a (complex) Fourier series approach. No quantitative evaluation is made although the authors claim their method is well-suited to digital computation. The interior flow problem is presently being pursued using an approach analogous to that used here for the exterior flow problem.

In the present paper a simpler method is proposed which is the logical extension to the unsteady problem of the method used by Randall for the steady problem with one major innovation. This is to reduce the problem analytically to a single integration (in terms of the Laplace transformation variable) for the aerodynamic generalized forces. The integration is performed using standard complex variable techniques analogous to those employed by Randall. Within the framework of the original problem formulation, the solution is mathematically exact.

A portion of the extensive numerical results for the aerodynamic generalized forces which have been obtained to date by this method is presented. The physical significance of these results is discussed, particularly in relation to the aeroelastic stability of a cylindrical shell of finite length.

2. Problem formulation. A solution to the equation of linearized, unsteady, potential flow,

$$\nabla^2\varphi - \left(\frac{1}{a^2}\right)\left[\frac{\partial^2\varphi}{\partial t^2} + 2U\frac{\partial^2\varphi}{\partial x\,\partial t} + U^2\frac{\partial^2\varphi}{\partial x^2}\right] = 0, \tag{1}$$

is sought (for $M > 1$) subject to the boundary condition

$$\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = \left[U\frac{\partial W}{\partial x} + i\omega W\right]\cos n\theta e^{i\omega t} \quad \text{for} \quad 0 < x < L,$$
$$= 0 \quad \text{for} \quad x < 0, \tag{2}$$

and also satisfying appropriate conditions at infinity. In the above, φ is the velocity potential and

$$w(x, \theta, t) = W(x)\cos n\theta e^{i\omega t} \tag{3}$$

is the radial deflection of the midplane of the (thin) cylinder. The aerodynamic pressure on the cylinder may be determined from the well-known Bernoulli equation

$$p = -\rho\left[\frac{\partial\varphi}{\partial t} + U\frac{\partial\varphi}{\partial x}\right]_{r=R}. \tag{4}$$

Consistent with the assumed cylinder deflection is a velocity potential of the form

$$\varphi = \Phi(x, r)\cos n\theta e^{i\omega t}. \tag{5}$$

Substitution of Eq. (5) into Eqs. (1) and (2) gives

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} - \frac{n^2}{r^2}\Phi - \left(\frac{1}{a^2}\right)\left[-\omega^2\Phi + 2i\omega U\frac{\partial\Phi}{\partial x} + U^2\frac{\partial^2\Phi}{\partial x^2}\right] = 0, \tag{6}$$

$$\left.\frac{\partial\Phi}{\partial r}\right|_{r=R} = U\frac{\partial W}{\partial x} + i\omega W \quad \text{for} \quad 0 < x < L,$$
$$= 0 \quad \text{for} \quad x < 0. \tag{7}$$

Following Randall's solution for the steady flow problem (see Holt and Strack and also

Stearman for formulations of the unsteady problem) Eq. (6) is solved by utilizing a Laplace transform with respect to the streamwise variable, x, i.e.,

$$\Phi^*(s, r) = \int_0^\infty \Phi(x, r)e^{-sx}\, dx. \tag{8}$$

Eqs. (6) and (7) now become, respectively,

$$s^2\Phi^* + \frac{d^2\Phi^*}{dr^2} + \frac{1}{r}\frac{d\Phi^*}{dr} - \frac{n^2}{r^2}\Phi^* - \left(\frac{1}{a^2}\right)[-\omega^2 + 2i\omega Us + U^2s^2]\Phi^* = 0 \tag{9}$$

and

$$\left.\frac{d\Phi^*}{dr}\right|_{r=R} = [Us + i\omega]W^*. \tag{10}$$

In deriving the above, explicit use has been made of the conditions $\Phi = 0$, $\partial\Phi/\partial x = 0$, $w = 0$ for $x < 0$.

The solution to Eq. (9) subject to the boundary condition, Eq. (10), and satisfying the condition of "finiteness at infinity"* is

$$\Phi^*(s, r) = RU\frac{K_n(\zeta r/R)}{\zeta K_n'(\zeta)}(\bar{s} + ik)W^*, \tag{11}$$

where $\zeta^2 \equiv (R/L)^2[M^2(\bar{s} + ik)^2 - \bar{s}^2]$ and $\bar{s} \equiv sL$, $k \equiv \omega L/U$.

Utilizing the convolution and inversion theorems, one has

$$\Phi_{r=R} = \frac{RU}{L}\int_0^x \left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{K_n(\zeta)}{\zeta K_n'(\zeta)}e^{s(x-\xi)}\, ds\right]\left(\frac{\partial}{\partial\xi} + ik\right)W(\xi)\, d\xi, \tag{12}$$

where the bars over x and s have been dropped. From Eqs. (4) and (12) the pressure amplitude ($P = (p/\cos n\theta)e^{-i\omega t}$) may be computed as

$$\begin{aligned} P &= -\frac{\rho U}{L}\left(\frac{\partial}{\partial x} + ik\right)\Phi\bigg|_{r=R} \\ &= -\left(\frac{R}{L}\right)\rho U^2 \\ &\quad\cdot\left\{\int_0^x \left[\frac{\partial}{\partial(x-\xi)} + ik\right]F_n(x-\xi)\frac{W(\xi)}{L}\, d\xi + \frac{W(x)}{L}F_n'(0) + \left(\frac{\partial}{\partial x} + i2k\right)\frac{W(x)}{L}F_n(0)\right\}, \end{aligned} \tag{13}$$

where

$$F_n(x) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{K_n(\zeta)}{\zeta K_n'(\zeta)}e^{sx}\, ds.$$

Eq. (12) or (13) constitutes a *formal* solution to the problem. Equivalent formal solutions have been obtained by Holt and Strack (Eq. (12)) and also Stearman (Eq. (13)). In deriving Eq. (13) from Eq. (12) some care must be exercised since at $x = 0$, F_n, F_n' and F_n'' have the behavior of a step function, delta function and delta prime (doublet) function, respectively. This difficulty may be overcome by replacing x in Eq.(12) by $x - \epsilon$ where ϵ

*More precisely $K_n(\zeta r/R)$ is an analytic function which satisfies the radiation condition in the intervals $i\infty > s > s_1$ and $s_2 > s > -i\infty$ and the finiteness condition in the interval $s_1 > s > s_2$, where $s_1 = -iMk/(M + 1)$ and $s_2 = -iMk/(M - 1)$.

is a small positive parameter, performing the derivation of Eq. (13) and taking the limit as $\epsilon \to 0$. Also note that $F_n(x) \equiv 0$ for $x < 0$ and the more general function

$$G_n(x, r) \equiv \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_n(\zeta r/R)}{\zeta K_n'(\zeta)} e^{sx} \, ds$$

is zero for $x < B(r - R)$ as is to be expected on physical grounds since the uniform free-stream flow must be undisturbed before the first Mach line, $x - B(r - R) = 0$. To see this mathematically, let the above integral be evaluated by contour integration. For large s,

$$\frac{K_n(\zeta r/R)}{K_n'(\zeta)} e^{sx} \sim \left(\frac{r}{R}\right)^{1/2} \exp \{s[x - B(r - R)]\}.$$

When $x < B(r - R)$ the contour is closed in the right half s-plane and since there are no poles of $K_n'(\zeta)$ in the right half-plane the integral is zero.

The essential practical problem is, of course, the evaluation of the integrals arising in Eq. (13). For this, one may make use of Randall's results for the steady flow problem to evaluate (numerically) the integrals over s. It is pertinent to point out, however, that having done so, another quadrature over ξ is required to determine P. Also, in most aeroelastic applications one is concerned with certain weighted integrals (over x) of P so yet another integration is required. These weighted integrals of the aerodynamic pressure are the so-called aerodynamic generalized forces. That is, if

$$\frac{W(x)}{L} = \sum_m A_m \psi_m(x), \tag{14}$$

the (non-dimensional) aerodynamic generalized force, Q_{mr}, is

$$Q_{mr} \equiv \frac{\int_0^1 P_m(x) \psi_r(x) \, dx}{\rho U^2}, \tag{15}$$

where P_m is the aerodynamic pressure due to a deflection

$$\frac{W(x)}{L} = \psi_m(x).$$

A considerable economy of effort may be accomplished by dealing directly with Q_{mr} and bypassing the calculation of P (or Φ). In order to do this most efficiently, the integrations over ξ and x are carried out first and that over s is left to the end. After some algebraic manipulation one obtains from Eqs. (13) and (15),

$$Q_{mr} = \left(-\frac{R}{L}\right) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_n(\zeta)}{\zeta K_n'(\zeta)} G_{mr}(s) \, ds, \tag{16}$$

where

$$G_{mr}(s) = (s + ik)^2 \int_0^1 e^{s\eta} \left[\int_\eta^1 \psi_m(x - \eta) \psi_r(x) \, dx \right] d\eta$$
$$+ (s + 2ik) \int_0^1 \psi_m(x) \psi_r(x) \, dx + \int_0^1 \frac{d\psi_m(x)}{dx} \psi_r(x) \, dx. \tag{17}$$

Note that, in general, the quadratures required in Eq. (17) may be performed with relative ease for a known cylinder deflection. Often it will be possible to accomplish them analytically. When this is the case the problem is reduced to a single integration over s (Eq. (16)). This latter integration may be performed by standard mathematical techniques using the theory of complex variables. A similar integration has been performed by Randall for the steady flow case to compute the velocity potential influence function. The present calculation may be considered a generalization of his result. The details are given in Appendix A.

Finally, we remark that if the aerodynamic pressure is desired it may be computed from

$$P = \rho U^2 \sum_m \sum_r A_m Q_{mr} \psi_r(x), \tag{18}$$

where

$$\frac{W}{L} = \sum A_m \psi_m$$

is the cylinder deflection. Eq. (18) is derived from the fact that Q_{mr} may be considered the (generalized) Fourier coefficient of the aerodynamic pressure (Eq. (15)). For Eq. (18) to hold the functions $\psi_m(x)$ must form an orthogonal complete set over $0 < x < 1$. If the ψ_m are the natural vibration modes of the cylinder this will be the case, for example.

Before leaving the problem formulation, one interesting and useful analytical result should be mentioned. It can be shown that

$$Q_{mr} = (-1)^{m+r} Q_{rm} \tag{19}$$

under certain mild restrictions on ψ_m. These are that

(i) the ψ_m be orthogonal, i.e.,

$$\int_0^1 \psi_m \psi_r \, dx = 0 \qquad \text{if} \quad m \neq r,$$

(ii) the ψ_m be (alternatingly) symmetrical and antisymmetrical, i.e.,

$$\psi_m(x) = (-1)^{m+1} \psi_m(1 - x),$$

(iii) $$\psi_m(0) = \psi_m(1) = 0.$$

Again these conditions will be satisfied by the natural modes of a cylinder *if* the ends are restrained against radial deflection. A short derivation of Eq. (19) follows. From Eq. (16), Eq. (19) will hold if

$$G_{mr} = (-1)^{m+r} G_{rm}. \tag{20}$$

Eq. (20) will hold if each term comprising G_{mr} (see Eq. (17)) satisfies it. From condition (i) this is true of the second term of G_{mr}. By an integration by parts and use of conditions (ii) and (iii) this is true of the third term of G_{mr}, i.e.,

$$\int_0^1 \frac{d\psi_m}{dx} \psi_r \, dx = -\int_0^1 \frac{d\psi_r}{dx} \psi_m \, dx.$$

If $m + r$ is even, then one is integrating an even function times an odd and thus

$$\int_0^1 \frac{d\psi_m}{dx} \psi_r \, dx = -\int_0^1 \frac{d\psi_r}{dx} \psi_m \, dx = 0.$$

Conversely if $m + r$ is odd, then one is integrating an odd times an odd or an even times an even function and

$$\int_0^1 \frac{d\psi_m}{dx} \psi_r \, dx = -\int_0^1 \frac{d\psi_r}{dx} \psi_m \, dx \neq 0.$$

Finally, consider the first term comprising G_{mr}. It will satisfy Eq. (20) if

$$\int_\eta^1 \psi_m(x - \eta)\psi_r(x) \, dx = (-1)^{m+r} \int_\eta^1 \psi_r(x - \eta)\psi_m(x) \, dx.$$

Let $x - \eta = \xi$, then

$$\int_\eta^1 \psi_m(x - \eta)\psi_r(x) \, dx = -\int_0^{\eta-1} \psi_m(-\xi)\psi_r(-\xi + \eta) \, d\xi.$$

In view of (ii), the above integral becomes

$$-(-1)^{m+r} \int_0^{\eta-1} \psi_m(1 + \xi)\psi_r(1 + \xi - \eta) \, d\xi.$$

Let $\zeta = 1 + \xi$, then the integral equals

$$(-1)^{m+r} \int_\eta^1 \psi_m(\zeta)\psi_r(\zeta - \eta) \, d\zeta, \qquad \text{q.e.d.}$$

A proof of this result under considerably more restrictive conditions has been given by Miles [7].

Finally it is noted that, for steady flow, the Q_{mr} obey a similarity law

$$BQ_{mr} = BQ_{mr}(BR/L, n, m, r). \tag{21}$$

This is analogous to the usual result for lift or moment for wings.

3. Numerical results. Numerical results have been obtained for a particular family of cylindrical deflections

$$\psi_m(x) = \sin m\pi x, \qquad m = 1, 2, \cdots,$$

for various values of the cylinder and flow parameters. It is noted that the generalized aerodynamic forces are functions of six parameters

$$Q_{mr} = Q_{mr}(M, k, R/L, n, m, r).$$

Certain limiting cases are of special interest as well as being useful and these are briefly considered here before passing onto the numerical results.

(i) $R/L \to \infty$, two-dimensional, planar flow. This limiting case needs no further elaboration as it is well known.

(ii) $R/L \to 0$, $n \neq 0$, "Slender body" flow. From Eqs. (11) and (13), as $R/L \to 0$, $\zeta \to 0$, and $K_n(\zeta)/\zeta K_n'(\zeta) \to -1/n$* and thus

$$P \to \frac{\rho U^2}{n} \frac{R}{L} \left[\frac{d}{dx} + ik \right]^2 \frac{W}{L} \tag{22}$$

*See Miles [8] for further terms in this asymptotic representation of $K_n(\zeta)/\zeta K_n'(\zeta)$.

and from Eq. (15)

$$Q_{mr} \to \left(\frac{R}{L}\right)\frac{1}{n}\left\{\int_0^1 \psi_m''\psi_r\,dx + 2ik\int_0^1 \psi_m'\psi_r\,dx - k^2\int_0^1 \psi_m\psi_r\,dx\right\}. \tag{23}$$

Eqs. (22) and (23) will be taken as the definitions of $P_{\text{slender body}}$ and $Q_{mr\ \text{slender body}}$. (There is a slender body limit for $n = 0$ also, but it does not have the simple form of Eq. (22) or (23)).

(iii) For steady flow, $k = 0$, R/L may be replaced by BR/L in (i) and (ii), recall Eq. (21). Thus n, m and r are fixed as the limit is taken, but not M. As a practical matter the "slender body" result is the limit as $BR/L \to 0$ even for $k \neq 0$ as long as $k < 1$ say.

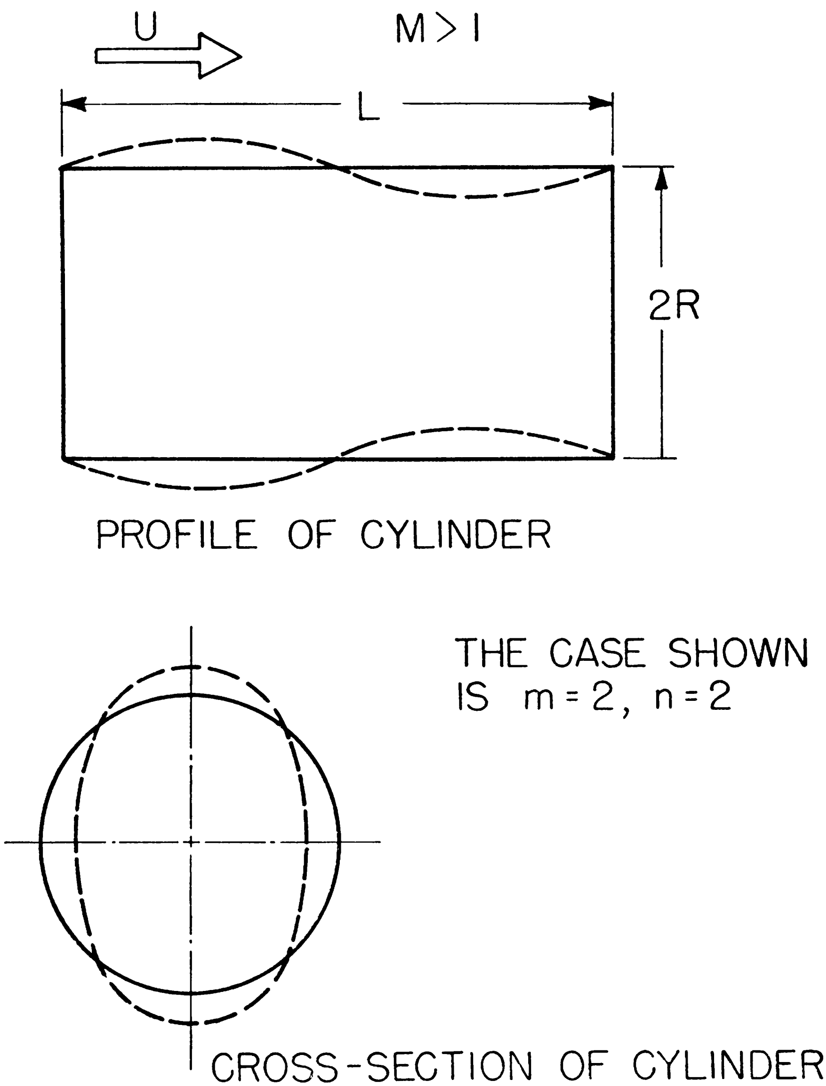

Fig. 1. Cylinder Geometry.

(iv) For $M \gg 1$, the two-dimensional limit may be further simplified to the well-known quasi-steady result

$$P = \frac{\rho U^2}{(M^2 - 1)^{1/2}} \left[\frac{\partial (W/L)}{\partial x} + \frac{M^2 - 2}{M^2 - 1} ik \frac{W}{L} \right]. \tag{24}$$

Again from Eq. (15)

$$Q_{mr} = \frac{1}{(M^2 - 1)^{1/2}} \int_0^1 \psi'_m \psi_r \, dx + \frac{M^2 - 2}{(M^2 - 1)^{3/2}} ik \int_0^1 \psi_m \psi_r \, dx. \tag{25}$$

Note that the "slender-body" limit is inherently quasi-steady (Eq. (22) or (23)), i.e., a finite power series in ω or k.

(v) As $n \to \infty$ the "slender body" theory is also the appropriate limit* and, on physical grounds, it may be argued that it is the appropriate limit whenever

$$\frac{BR(m \text{ or } r)}{Ln} \ll 1,$$

where the larger of the two integers, m or r, is to be used. The parameter on the left is the ratio of circumferential to streamwise wavelength (within a constant of π) modified by B by analogy to the similarity variable for steady flow. It is to be emphasized that this single parameter characterizes the regime in which the "slender body" limit is accurate and that its "derivation" is based on physical reasoning. For fixed m, r, n and $k = 0$, the parameter reduces to a known mathematical result (ii) and (iii) and no numerical results obtained to date have invalidated its use for defining the limit of applicability of the "slender body" approximation.

A few typical numerical results will be now presented. More extensive numerical results will be published elsewhere. First consider Fig. 2. Here Q_{11_R} is plotted versus

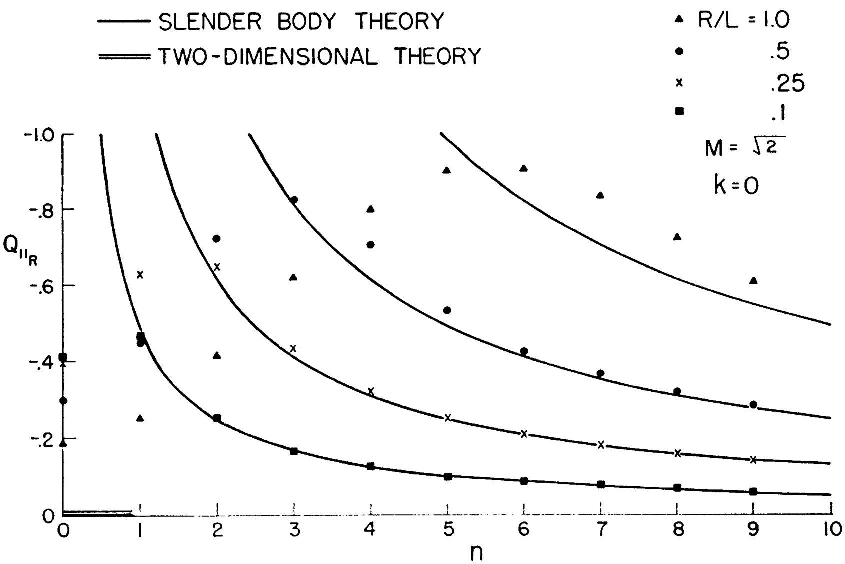

FIG. 2. Generalized Aerodynamic Force vs. Circumferential Mode Number.

*$K_n(\zeta)/\zeta K'_n(\zeta) \to -1/n$ as $n \to \infty$.

n for $M = 2^{1/2}$, $k = 0$ and several R/L. (Recall the similarity for steady flow, Eq. (21); the results shown may be extended to other R/L and M for the same BR/L.) The solid lines are the "slender body" limits. Of course, n can take on only integer values; the solid lines are used to make the comparison between the "exact" results and the "slender body" limit easier. As may be seen for each R/L the "exact" results asymptotically approach the "slender body" limit for sufficiently large n. The limit is reached for a smaller value of n as R/L decreases. The two-dimensional limit is that $Q_{11_R} = 0$. It may be seen that this limit is being approached for small n (particularly $n = 0$) as R/L increases. Similar results have been obtained for Q_{22_R}, Q_{33_R}, etc. As is to be expected, the shorter streamwise wavelength (larger $m/2L$) delays the approach to the "slender body" limit to higher n for a given R/L. A coupling generalized force Q_{12_R} is shown in Fig. 3. For small n the "exact" results are approximately equal to the two-dimensional limit (for Q_{12_R}, this limit is non-zero) for the larger R/L. For all R/L, Q_{12_R} falls to zero, the "slender body" limit, as n increases. The drop to zero occurs for smaller n as R/L decreases as is to be expected.

Next, unsteady flow results are considered. The effect of $k \neq 0$ on the real parts of the generalized forces is usually small for $k < 1$, and therefore the imaginary parts are emphasized. In Fig. 4, Q_{11_I} is plotted* versus n for several R/L and for $k = 1.0$. As n increases, Q_{11_I} falls to zero, which is the "slender body" limit. The effect of R/L is qualitatively the same as for Q_{11_R}. The two-dimensional limit is not known in a simple analytical form for $k \neq 0$, except for $M \gg 1$, and thus the small n, large R/L (i.e., two-dimensional) limit has not been determined.

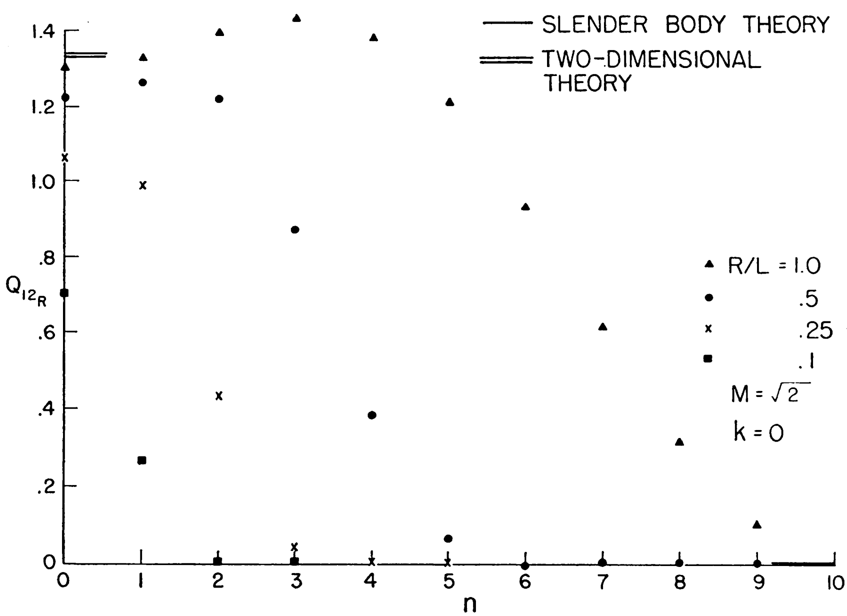

FIG. 3. Generalized Aerodynamic Force vs. Circumferential Mode Number.

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*Q_{mr}, $m = 1, \cdots, 4$, $r = 1, \cdots, 4$ have been computed for several k for various R/L and $n = 0 \rightarrow 9$. Only selected results are presented here for the sake of brevity.

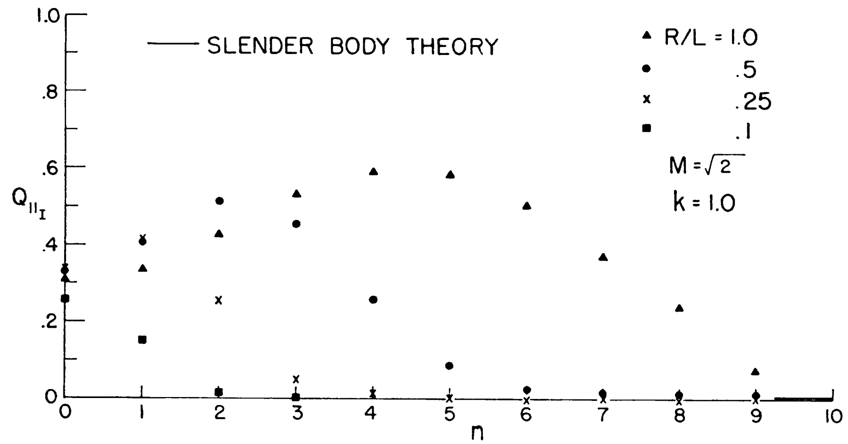

Fig. 4. Generalized Aerodynamic Force vs. Circumferential Mode Number.

The last two figures are concerned with Q_{11_I} at low supersonic Mach numbers and various R/L. It is well known that, for two-dimensional flow over a flat plate at sufficiently low Mach number, Q_{11_I} can become negative. When Q_{11_I} is less than zero, the $m = 1$ mode can act as a single-degree-of-freedom oscillator with negative damping, and the system may be unstable. A plot of Q_{11} versus k is shown in Fig. 5 for $R/L = 1.0$, $n = 0$ and several M's. For sufficiently large M, Q_{11_I} is always positive. For $M < 1.3$,

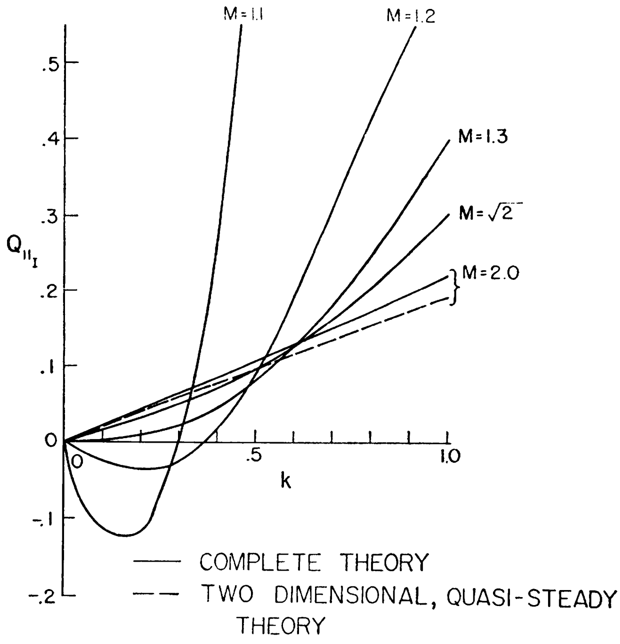

Fig. 5. Generalized Aerodynamic Force vs. Reduced Frequency.

Q_{11_I} is negative over a portion of the range of reduced frequency. Similar results have been obtained for other values of R/L. As R/L decreases the highest Mach number for which Q_{11_I} can be negative also decreases. A plot of the locus of such Mach numbers is shown in Fig. 6. For $1 < M < M_{\text{locus}}$ single-degree-of-freedom flutter is possible for $n = 0$. A similar curve for $n = 1$ is also shown. Note that the region for which single degree-of-freedom flutter is possible is much smaller for $n = 1$ than for $n = 0$. Apparently the three-dimensionality of the flow tends to suppress "negative aerodynamic damping" (i.e., $Q_{11_I} < 0$). The trend with n is evident.

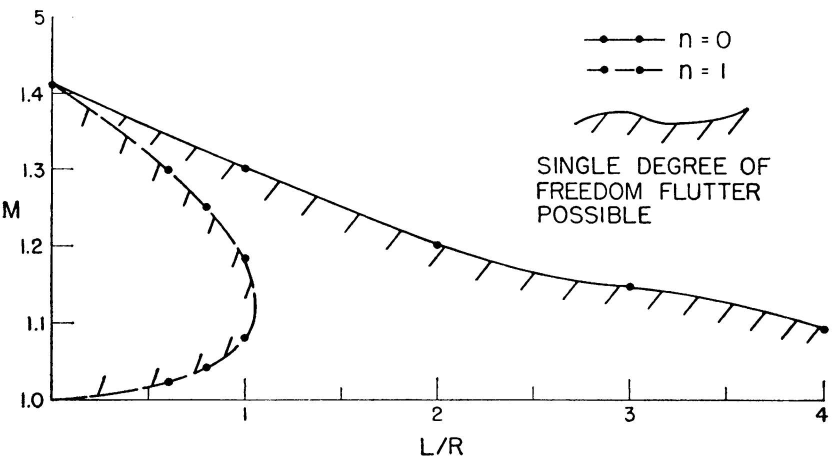

FIG. 6. Mach Number vs. Length to Radius Ratio.

4. Qualitative discussion of the aeroelastic stability of a cylindrical shell of finite length. It is, of course, very desirable to perform a systematic stability investigation of a cylindrical shell using the present aerodynamic forces. However, certain important qualitative features of the stability problem may be determined without making detailed stability calculations. One of these was discussed in the previous section where the Mach number, length-to-radius ratio domain was determined wherein single-degree-of-freedom flutter is possible. By analogy to the flat plate problem and the agreement found herein between the "exact" aerodynamic theory and two-dimensional quasi-steady theory for small n, large m, and moderate to large R/L; one may expect that the classical "coalescence" type of flutter may also occur for cylindrical shells for the afore-mentioned range of geometrical, flow, and modal parameters. It is reasonable to ask next if there is a characteristic type of instability associated with the parameter range wherein the "slender body" type of aerodynamic approximation is valid. Consider Eq. (22) for the aerodynamic pressure

$$P_{\text{slender body}} = \rho \frac{U^2 R}{nL^2} \left[\frac{d^2 W}{dx^2} + 2ik \frac{dW}{dx} - k^2 W \right].$$

The last term on the right-hand side is of the virtual mass type and will not be important for typical values of air/shell mass ratio. It cannot lead to an instability. The second term provides positive damping and cannot lead to an instability. It too will not be of great importance for most aeronautical applications. However, the first term is the aerodynamic analogue to an in-plane axial compressive force and can lead to a "buckling" or "divergence" type of instability. (The aerodynamic damping term will modify this slightly but not greatly.) A "flutter" or "oscillatory" type of instability may also occur but only subsequent to the occurrence of aerodynamic buckling, i.e., at higher velocities. Detailed stability studies using "slender body" theory will be published elsewhere. There is an interesting analogy between the above problem and that of a low aspect ratio flat plate with free side edges. The low aspect ratio membrane problem [9] should also be mentioned though it has certain special features which lead to somewhat pathological stability behavior. Finally, the traveling wave flutter of shells of large L/R for the $n = 0$ mode is yet another distinct type of instability [8; 10].

To fully investigate the aeroelastic stability of a cylindrical shell all of the above types of instability need to be considered. All can be considered with the present aerodynamic theory combined with an appropriate shell theory. Undoubtedly, simplifications are possible for certain combinations of M, L/R and n, some of which have been discussed above. The use of the present aerodynamic theory in a systematic stability investigation will permit an evaluation of the accuracy of these simplified theories as well as perhaps suggest others.

5. Conclusions. A method has been developed to compute the generalized aerodynamic forces on an harmonically oscillating cylindrical shell within the framework of classical, linearized, potential flow theory*. The present method is thought to be more accurate and more efficient than previously suggested methods and is the only one to date which has been carried through to the point of obtaining quantitative results. Quantitative results are shown in sufficient number to demonstrate the nature of aerodynamic forces and various limiting cases have been identified where the aerodynamic forces take on a simpler form. The most useful of these limiting cases appears to be the "slender body" limit which is the limit as

$$\frac{BRm}{Ln} \to 0.$$

In particular it is the appropriate limit, for fixed shell and flow parameters, as the circumferential mode number becomes large.

A brief qualitative discussion of the types of instability to be expected for a cylindrical shell has been given based on the nature of aerodynamic forces. The region where single-degree-of-freedom flutter is possible has been identified and it has been pointed out that in the "slender body" limit an "aerodynamic buckling" type of instability can occur.

Certain extensions of the present analysis may prove desirable. Two of these are:

(i) The extension to subsonic flow. The downstream influence may make the physical model less realistic for subsonic flow, however.

(ii) The extension to other than simple harmonic motion, in particular to unstable motion.

Both of these would appear relatively straightforward. For (i) the use of a Fourier rather than a Laplace transformation integration variable is necessary. These extensions are currently under investigation.

*The flow is taken to be supersonic.

Appendix A

Evaluation of the Aerodynamic Integral

Here the evaluation of the integral (Eq. (16))

$$Q_{mr} = \frac{-(R/L)}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_n(\zeta)}{\zeta K_n'(\zeta)} G_{mr}(s)\, ds \tag{A-1}$$

is considered. The evaluation will be accomplished by contour integration in the (complex) s-plane.

The relation between ζ and s,

$$\zeta = (R/L)[M^2(s + ik)^2 - s^2]^{1/2}, \tag{A-2}$$

is made unique by placing a branch line between $s = s_1 = -iMk/(M + 1)$ and $s = s_2 = -iMk/(M - 1)$.

(For steady flow, $k = 0$, Eq. (A-2) reduces to

$$\zeta = (R/L)(M^2 - 1)^{1/2} s$$

and this branch line is no longer required.)

In addition one must select the branch of the function $K_n(\zeta)$. The branch line is taken along the negative real ζ-axis. In the s-plane this forms a T-cut between $s = s_1$, $s = s_2$ and

$$s = s_0 = -\frac{iM^2 k}{M^2 - 1}, \qquad s = -\infty + s_0$$

(see Fig. A.1). Therefore the contour of integration is chosen as shown in Fig. A.1. In the usual manner the radius of the semi-circle is allowed to approach infinity and it may be shown that the contribution to the line integral along the arc at infinity is zero. Thus,

$$\frac{-(R/L)}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_n(\zeta)}{\zeta K_n'(\zeta)} G_{mr}(s)\, ds = -(R/L) \sum_j \frac{K_n(\zeta_j) G_{mr}(s_j)}{(d/d\zeta)(\zeta K_n'(\zeta))(d\zeta/ds)\mid_{\zeta=\zeta_j}}$$

$$\cdot \frac{-(R/L)}{2\pi i} \left\{ \int_{-\infty+s_0}^{s_0} \frac{K_n(re^{-i\pi})}{re^{-i\pi} K_n'(re^{-i\pi})} G_{mr}\, ds + \int_{s_0}^{s_2} (-i\pi) \cdots ds + \int_{s_2}^{s_0} (0) \cdots ds \right. \tag{A-3}$$

$$\left. + \int_{s_0}^{s_1} (0) \cdots ds + \int_{s_1}^{s_0} (i\pi) \cdots ds + \int_{s_0}^{-\infty+s_0} (i\pi) \cdots ds \right\},$$

where r is the modulus and ( ) the argument of ζ along the appropriate branch and where ζ_j are the zeros of K_n' and the corresponding s_j are determined by solving Eq. (A-2):

$$s_j = -\frac{ikM^2}{M^2 - 1} + \frac{[\zeta_j^2 (L/R)^2 (M^2 - 1) - k^2 M^2]^{1/2}}{M^2 - 1}.$$

The square root is chosen so that $s_j + ikM^2/(M^2 - 1)$ and ζ_j are of the same sign. Implicit in the above is the assumption that G_{mr} does not contribute to the sum of the residues. This would appear to be the usual case. If G_{mr} does contribute, it must be accounted for in summing the residues. For the example problem considered in the text, G_{mr} does have poles, however the residues at these poles are zero, i.e., there are no first-order poles. Randall has tabulated the ζ_j for $n = 1, \cdots, 10$ (see

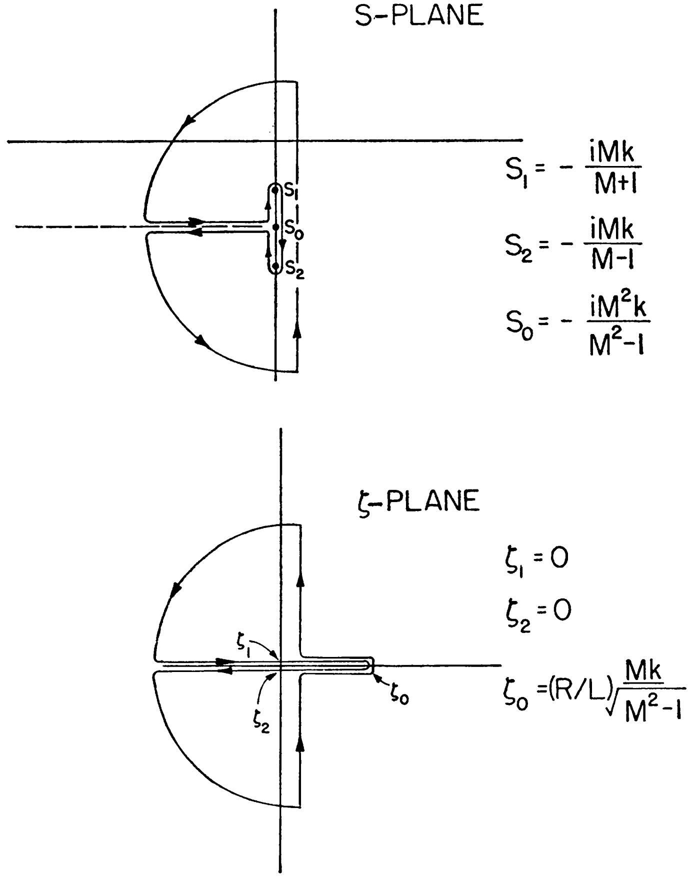

Fig. A.1. Integration Contour.

Table 2, reference [3]). Quoting Randall, "The general result, for the zeros of $K_n'(\zeta)$, is that $K_n'(\zeta)$ has all its zeros to the left of the imaginary axis and the number of zeros is the nearest even integer to $n + \frac{1}{2}$. The zeros are symmetrically placed about the real (ζ)-axis $\cdots$." Also see Watson [11].

Eq. (A-3) may be further simplified to bring it into a form more convenient for computation. Consider first the summation representing the contribution of the residues.

$$-\left(\frac{R}{L}\right) \sum_i \frac{K_n(\zeta_i) G_{mr}(s_i)}{(d/d\zeta)(\zeta K_n'(\zeta))(d\zeta/ds) \,|_{\substack{\zeta = \zeta_i \\ s = s_i}}}.$$

Now

$$\frac{d}{d\zeta}(\zeta K_n'(\zeta)) = \zeta K_n'' + K_n'$$

thus

$$\frac{d}{d\zeta}(\zeta K_n') \frac{d\zeta}{ds} = \zeta \frac{d\zeta}{ds} K_n'' + K_n' \frac{d\zeta}{ds} = \frac{1}{2} \frac{d(\zeta^2)}{ds} K_n'' + K_n' \frac{d\zeta}{ds};$$

at

$$\zeta = \zeta_i\,, \qquad s = s_i\,, \quad \text{it equals}$$

$$\left(\frac{R}{L}\right)^2 [s_i(M^2 - 1) + ikM^2]K_n''$$

but

$$K_n''(\zeta) = -\frac{1}{\zeta} K_n'(\zeta) + (1 + n^2/\zeta^2)K_n(\zeta);$$

at

$$\zeta = \zeta_i\,, \quad \text{it equals}$$

$$(1 + n^2/\zeta_i^2)K_n(\zeta_i).$$

Therefore

$$-\left(\frac{R}{L}\right) \sum_i \frac{K_n(\zeta_i)G_{mr}(s_i)}{(d/d\zeta)(\zeta K_n'(\zeta))(d\zeta/ds)\,|_{\substack{\zeta=\zeta_i\\ s=s_i}}} = \frac{-1}{(R/L)} \sum_i \frac{G_{mr}(s_i)\zeta_i^2}{[s_i(M^2 - 1) + ikM^2][\zeta_i^2 + n^2]}. \tag{A-4}$$

Now consider the integrals; these may be combined as follows:

$$\frac{-(R/L)}{2\pi i}\left\{\int_{-\infty+s_0}^{s_0} G_{mr}\left[\frac{K_n(re^{-i\pi})}{re^{-i\pi}K_n'(re^{-i\pi})} - \frac{K_n(re^{i\pi})}{re^{i\pi}K_n'(re^{i\pi})}\right] ds\right.$$
$$\left. + \int_{s_1}^{s_0} G_{mr}\left[\frac{K_n(re^{i\pi})}{re^{i\pi}K_n'(re^{i\pi})} - \frac{K_n(r)}{rK_n'(r)}\right] ds + \int_{s_0}^{s_2} G_{mr}\left[\frac{K_n(re^{-i\pi})}{re^{-i\pi}K_n'(re^{-i\pi})} - \frac{K_n(r)}{rK_n'(r)}\right] ds\right\}. \tag{A-5}$$

Thus the final form of Q_{mr} used for computational purposes is

$$Q_{mr} = \frac{-1}{(R/L)} \sum_i \frac{G_{mr}(s_i)\zeta_i^2}{[s_i(M^2 - 1) + ikM^2][\zeta_i^2 + n^2]} \frac{-(R/L)}{2\pi i}$$
$$\cdot\left\{\int_{-\infty+s_0}^{s_0} G_{mr}\left[\frac{K_n(re^{-i\pi})}{re^{-i\pi}K_n'(re^{-i\pi})} - \frac{K_n(re^{i\pi})}{re^{i\pi}K_n'(re^{i\pi})}\right] ds\right. \tag{A-6}$$
$$\left. + \int_{s_1}^{s_0} G_{mr}\left[\frac{K_n(re^{i\pi})}{re^{i\pi}K_n'(re^{i\pi})} - \frac{K_n(r)}{rK_n'(r)}\right] ds + \int_{s_0}^{s_2} G_{mr}\left[\frac{K_n(re^{-i\pi})}{re^{-i\pi}K_n'(re^{-i\pi})} - \frac{K_n(r)}{rK_n'(r)}\right] ds\right\}.$$

Perhaps a final word is in order with regard to the use of contour integration for the evaluation of the integral. The calculation could have similarly been carried out by making the transformation, $\alpha = is$. (α may be thought of as a Fourier transform variable.) Utilizing the α variable, the integral for Q_{mr} (Eq. A-1)) becomes one along the real axis (in the α-plane) which can be evaluated by a (wholly) numerical integration, for example. It is thought the present method is somewhat more advantageous for supersonic flow. The wholly numerical integration approach has also been programmed for machine computation and it has been found that for a desired degree of accuracy, this approach requires considerably more computation time than the contour integration method, approximately a factor of ten. This is because in the present method the residue contribution dominates the integral contribution, except for $n = 0$, and also the integrand of the integral decays exponentially along the branch line to infinity.

APPENDIX B

THE FORM OF G_{mr} FOR $\psi_m = \sin m\pi x$

There are two cases to be considered depending on whether m and r are or are not equal.

(i) $m = r$:

$$G_{mm}(s) = \frac{(s+ik)^2}{2}\left\{\frac{-s}{s^2+(m\pi)^2} + \frac{2(m\pi)^2}{[s^2+(m\pi)^2]^2}\,[1-(-1)^m e^s]\right\} + \frac{(s+2ik)}{2}.$$

(ii) $m \neq r$:

$$G_{mr} = (s+ik)^2\left[\frac{rm}{r^2-m^2}\right]\left\{\frac{-1}{(r\pi)^2+s^2}\,[1-(-1)^r e^s] + \frac{(-1)^{m+r}}{(m\pi)^2+s^2}\,[1-(-1)^m e^s]\right\}$$
$$+\,[1-(-1)^{m+r}]\left[\frac{rm}{r^2-m^2}\right].$$

Note that G_{mm} has poles at $s = \pm im\pi$ and G_{mr} $(m \neq r)$ has poles at $s = \pm im\pi$, $s = \pm ir\pi$. However, the residues at these poles are zero. Also note that $G_{mr} = (-1)^{m+r} G_{rm}$ as is to be expected from the discussion at the end of Section 2.

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