STRESS FUNCTIONS FOR COUPLE AND DIPOLAR STRESSES*

By D. E. CARLSON, (*University of Illinois, Urbana, Illinois*)

1. Introduction. In a recent paper Gurtin [5] established a complete representation for symmetric second-order tensor fields in terms of another symmetric second-order tensor field and a vector field. His representation is analogous to the Stokes-Helmholtz resolution of a vector field into the curl of another vector field and the gradient of a scalar field. Gurtin was thus led to a complete representation for equilibrated stress fields in continuum mechanics—the generalized Beltrami solution. It is the purpose of the present paper to extend these results to more general theories of continuum mechanics, namely the Cosserat [1] continuum and the dipolar continuum of Green and Rivlin [6].

Günther [3] has obtained a stress function solution of the Cosserat equations of equilibrium. However, his solution cannot be complete for regions with holes because in the absence of couple stresses, it reduces to the Beltrami solution (cf. Gurtin [5]).

Sections 2 and 3 contain the equilibrium equations of the above theories and several solutions in terms of stress functions. In Section 4, following Gurtin [5], we establish some general representation theorems for second- and third-order tensor fields and utilize them to demonstrate the completeness of the solutions presented in Sections 2 and 3.

2. Couple stresses. The stress equations of equilibrium for a Cosserat [1] continuum may be written as

$$t_{ij,i} + f_i = 0,$$

$$m_{ij,i} + e_{ikl}t_{kl} + c_i = 0$$

(2.1)

when referred to rectangular Cartesian coordinates.** In Eqs. (2.1) t_{ij} is the stress tensor, m_{ij} the couple stress tensor, f_i the body force per unit volume, and c_i the body couple per unit volume.

Introducing the quantities

$$m_{ijk} = \tfrac{1}{2}e_{ljk}m_{il} \qquad (m_{ij} = e_{imn}m_{imn}),$$

$$c_{ij} = \tfrac{1}{2}e_{lij}c_l \qquad (c_i = e_{imn}c_{mn}),$$

(2.2)

we can write Eqs. (2.1) in the equivalent form

$$t_{ij,i} + f_i = 0,$$

$$m_{ijk,i} + t_{[jk]} + c_{jk} = 0.$$

(2.3)

Note from (2.2) that

$$m_{i(jk)} = 0, \qquad c_{(ij)} = 0.$$

(2.4)

**We use the usual indicial notation of Cartesian tensor analysis. Latin subscripts have the range (1, 2, 3), and summation over repeated subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate. Kronecker's delta and the alternating symbol are denoted by δ_{ij} and e_{ijk}, respectively. A pair of subscripts is enclosed by parentheses ( ) to indicate the symmetric part and by brackets [ ] to indicate the skewsymmetric part.

We will work with m_{ij} and Eqs. (2.1). However, we can immediately write our results in terms of m_{ijk} by use of Eqs. (2.2).

The following theorems, which provide solutions to Eqs. (2.1), may be confirmed by direct substitution. We choose not to include the obvious smoothness requirements in the statements of these theorems.

THEOREM 2.1.   *Let ϑ_i , φ_i , θ_i , and ϕ_i satisfy*

$$\nabla^2 \vartheta_i + e_{ijk}\varphi_{j,k} = -f_i \ ,$$
$$\nabla^2 \theta_i + e_{ijk}\phi_{j,k} + 2\varphi_i = -c_i \ . \tag{2.5}$$

Define t_{ij} and m_{ij} by

$$t_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} + e_{ijk}\varphi_k \ ,$$
$$m_{ij} = e_{imp}e_{jnq}\Omega_{mn,pq} - \delta_{ij}\theta_{k,k} + \theta_{i,j} + \theta_{j,i} + e_{ijk}\phi_k \ , \tag{2.6}$$

where $\omega_{ii} = \omega_{ij}$ and $\Omega_{ii} = \Omega_{ij}$. Then t_{ij} and m_{ij} satisfy Eqs. (2.1).

In particular, if c_i , φ_i , θ_i , ϕ_i , and Ω_{ij} vanish identically, we recover Schaefer's [2] solution of the usual equations of continuum mechanics (cf. footnote in Section 4).

THEOREM 2.2.   *Let ϑ_i , φ_i , and η_i satisfy*

$$\nabla^2 \vartheta_i + e_{ijk}\varphi_{j,k} = -f_i \ ,$$
$$\nabla^2 \eta_i + 2\varphi_i = -c_i \ . \tag{2.7}$$

Define t_{ij} and m_{ij} by

$$t_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} + e_{ijk}\varphi_k \ ,$$
$$m_{ij} = e_{imp}e_{jnq}\chi_{mn,pq} - \delta_{ij}\sigma_{k,k} + \sigma_{i,j} + \eta_{j,i} \ , \tag{2.8}$$

where here $\omega_{ii} = \omega_{ij}$ but χ_{ij} need not be symmetric and σ_i is arbitrary. Then t_{ij} and m_{ij} satisfy Eqs. (2.1).

The generality of the above solutions will be established in Section 4.

3. Dipolar stresses.   In the dipolar case of the multipolar continuum mechanics of Green and Rivlin [6], the equilibrium equations appear as

$$T_{ij,i} + F_j = 0, $$
$$M_{i[jk],i} + T_{[jk]} + C_{[jk]} = 0. \tag{3.1}$$

In Eqs. (3.1) T_{ij} is the simple surface stress monopole, M_{ijk} the simple surface stress dipole of the first kind, F_i the simple body force monopole per unit volume, and C_{ij} the simple body force dipole of the first kind per unit volume.

We note that, in light of Eqs. (2.4), Eqs. (3.1) have the same form as Eqs. (2.3). Hence for each of our results pertaining to t_{ij} and m_{ijk} an analogous result pertains to T_{ij} and $M_{i[jk]}$. In particular, Eqs. (3.1) are equivalent to

$$T_{ij,i} + F_j = 0, $$
$$M_{ij,i} + e_{jkl}T_{kl} + C_j = 0, \tag{3.2}$$

where

$$M_{ij} = e_{jmn}M_{imn} \qquad (M_{i[jk]} = \tfrac{1}{2}e_{ljk}M_{il}), $$
$$C_i = e_{imn}C_{mn} \qquad (C_{[ij]} = \tfrac{1}{2}e_{lij}C_l). \tag{3.3}$$

The following theorem, which may be confirmed by substitution, provides an alternative solution in terms of M_{ijk} and not just $M_{i[jk]}$. It will be shown to be complete in Section 4.

THEOREM 3.1.   *Let ϑ_i , φ_i , and ψ_{ij} satisfy*

$$\nabla^2\vartheta_i + e_{ijk}\varphi_{j,k} = -F_i ,$$
$$\nabla^2\psi_{[ij]} + e_{ijk}\varphi_k = -C_{[ij]} . \tag{3.4}$$

Define T_{ij} and M_{ijk} by

$$T_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} + e_{ijk}\varphi_k ,$$
$$M_{ijk} = e_{ilp}e_{jmq}e_{knr}\Lambda_{lmn,pqr} + \gamma_{lm,lmijk} - \nabla^2(\gamma_{il,ljk} + \gamma_{lj,lik}) + \nabla^4\gamma_{ij,k} \tag{3.5}$$
$$- \lambda_{lk,lij} + \nabla^2\lambda_{ik,j} + \psi_{jk,i} ,$$

where $\omega_{ii} = \omega_{ij}$ but Λ_{ijk} , γ_{ij} , and λ_{ij} are arbitrary. Then T_{ij} and M_{ijk} satisfy Eqs. (3.1).

If we represent the skewsymmetric tensors $\psi_{[ij]}$ and $C_{[ij]}$ in terms of their dual vectors

$$\psi_i = e_{imn}\psi_{mn} \qquad (\psi_{[ij]} = \tfrac{1}{2}e_{lij}\psi_l),$$
$$C_i = e_{imn}C_{mn} \qquad (C_{[ij]} = \tfrac{1}{2}e_{lij}C_l), \tag{3.6}$$

Eqs. (3.4) become

$$\nabla^2\vartheta_i + e_{ijk}\varphi_{j,k} = -F_i ,$$
$$\nabla^2\psi_i + 2\varphi_i = -C_i , \tag{3.7}$$

and these equations are identical with Eqs. (2.7).

4. Completeness theorems.   For the remainder of the paper R will denote a bounded, open, integrable (Jordan-measurable) region of a three-dimensional Euclidean space. The closure of R will be denoted by R'. We write $f \,\varepsilon\, C^N(R)$ if and only if f is a real-valued function defined and N times continuously differentiable on R, and we write $f \,\varepsilon\, C^0(R')$ if and only if f is defined and continuous on R'.

The proofs of our representation theorems will be based in part on the following lemmas. A proof of the first lemma can be found in Courant [4, p. 246]. The latter lemmas follow successively from the first one.

LEMMA 4.1.   *Let $\mu \,\varepsilon\, C^0(R')$ and $\mu \,\varepsilon\, C^N(R)$ with $N \geq 1$. Define f by*

$$f(\mathbf{x}) = -\frac{1}{4\pi}\int_R \frac{\mu(\xi)}{|\mathbf{x} - \xi|}\,d\xi. \tag{4.1}$$

Then $f \,\varepsilon\, C^{N+1}(R)$ and $\nabla^2 f = \mu$ on R.

LEMMA 4.2.   *Let the hypotheses of Lemma 4.1 hold but with $N \geq 2$, and define f by (4.1). Define g by*

$$g(\mathbf{x}) = -\frac{1}{4\pi}\int_R \frac{f(\xi)}{|\mathbf{x} - \xi|}\,d\xi. \tag{4.2}$$

Then $g \,\varepsilon\, C^{N+2}(R)$ and $\nabla^4 g = \mu$ on R.

LEMMA 4.3.   *Let the hypotheses of Lemma 4.1 hold but with $N \geq 3$, and define f and g*

by (4.1) *and* (4.2), *respectively. Define h by*

$$h(\mathbf{x}) = -\frac{1}{4\pi} \int_R \frac{g(\xi)}{|\mathbf{x} - \xi|} \, d\xi. \tag{4.3}$$

Then $h \ \varepsilon \ C^{N+3}(R)$ *and* $\nabla^6 h = \mu$ *on* R.

We now turn to the statements and proofs of our completeness theorems.

THEOREM 4.1. *Let* $t_{(ij)} \ \varepsilon \ C^0(R')$, $t_{(ij)} \ \varepsilon \ C^N(R)$, *and* $m_{(ij)} \ \varepsilon \ C^0(R')$, $m_{(ij)} \ \varepsilon \ C^N(R)$ *with* $N \geq 2$. *Then there exist* $\omega_{ij} = \omega_{ji} \ \varepsilon \ C^N(R)$, $\vartheta_i \ \varepsilon \ C^{N-1}(R)$, φ_i, *and* $\Omega_{ij} = \Omega_{ji} \ \varepsilon \ C^N(R)$, $\theta_i \ \varepsilon \ C^{N-1}(R)$, ϕ_i *such that*

$$t_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{,j} + \vartheta_{j,i} + e_{ijk}\varphi_k \tag{4.4}$$

and

$$m_{ij} = e_{imp}e_{jnq}\Omega_{mn,pq} - \delta_{ij}\theta_{k,k} + \theta_{,j} + \theta_{j,i} + e_{ijk}\phi_k . \tag{4.5}$$

Moreover, if $N \geq 3$, $t_{[ij]} \ \varepsilon \ C^1(R)$, $m_{[ij]} \ \varepsilon \ C^1(R)$, *and* t_{ij} *and* m_{ij} *satisfy Eqs.* (2.1); *then*

$$\nabla^2 \vartheta_i + e_{ijk}\varphi_{j,k} = -f_i ,$$
$$\nabla^2 \theta_i + e_{ijk}\phi_{j,k} + 2\varphi_i = -c_i . \tag{4.6}$$

Proof. The second part of the theorem follows from Theorem 2.1. Since the representations of t_{ij} and m_{ij} are of the same form, it suffices to establish representation (4.4) for t_{ij}. We decompose t_{ij} into its symmetric and skewsymmetric parts:

$$t_{ij} = t_{(ij)} + t_{[ij]} . \tag{4.7}$$

Setting

$$\varphi_i = \tfrac{1}{2}e_{imn}t_{mn} , \tag{4.8}$$

we have

$$t_{[ij]} = e_{ijk}\varphi_k . \tag{4.9}$$

Turning now to the representation of $t_{(ij)}$, we note that by Lemma 4.2 there exist $A_{ij} = A_{ji} \ \varepsilon \ C^{N+2}(R)$ such that

$$t_{(ij)} = \nabla^4 A_{ij} . \tag{4.10}$$

Also, we have the identity

$$\nabla^4 A_{ij} = e_{ila}e_{jmb}e_{lpc}e_{mqd}A_{pq,abcd} - A_{lm,lmij} + \nabla^2(A_{il,lj} + A_{lj,li}), \tag{4.11}$$

as is readily verified with the aid of

$$e_{ijk}e_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp} . \tag{4.12}$$

Next define

$$\omega'_{lm} = e_{lpc}e_{mqd}A_{pq,cd} ,$$
$$\vartheta'_i = A_{il,l} , \tag{4.13}$$

and conclude from Eqs. (4.10) and (4.11) and the symmetry of A_{ij} that

$$t_{(ij)} = e_{imp}e_{jnq}\omega'_{mn,pq} - \vartheta'_{k,kij} + \nabla^2(\vartheta'_{i,j} + \vartheta'_{j,i}). \tag{4.14}$$

Now set

$$\Gamma_{ij} = \delta_{ij}\vartheta'_{k,k} \tag{4.15}$$

and note that

$$e_{imp}e_{jnq}\Gamma_{mn,pq} = \delta_{ij}\nabla^2\vartheta'_{k,k} - \vartheta'_{k,kij} . \tag{4.16}$$

Eqs. (4.14) and (4.16) yield

$$t_{(ij)} = e_{imp}e_{jnq}(\omega'_{mn,pq} + \Omega_{mn,pq}) - \delta_{ij}\nabla^2\vartheta'_{k,k} + \nabla^2(\vartheta'_{i,j} + \vartheta'_{j,i}). \tag{4.17}$$

The proof is completed by defining

$$\omega_{ij} = \omega'_{ij} + \Gamma_{ij} ,$$
$$\vartheta_i = \nabla^2\vartheta'_i , \tag{4.18}$$

so that

$$t_{(ij)} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} . \tag{4.19}$$

The representation (4.14) of the symmetric tensor $t_{(ij)}$ is due to Gurtin [5] as is the step which leads to the simpler representation (4.19).*

THEOREM 4.2. *Let $t_{(ij)}$ ε $C^0(R')$, $t_{(ij)}$ ε $C^N(R)$, and m_{ij} ε $C^0(R')$, m_{ij} ε $C^N(R)$ with $N \geq 2$. Then there exist $\omega_{ij} = \omega_{ji}$ ε $C^N(R)$, ϑ_i ε $C^{N-1}(R)$, φ_i, and χ_{ij} ε $C^N(R)$, σ_i ε $C^{N-1}(R)$, η_i ε $C^{N-1}(R)$ such that*

$$t_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} + e_{ijk}\varphi_k \tag{4.20}$$

and

$$m_{ij} = e_{imp}e_{jnq}\chi_{mn,pq} - \delta_{ij}\sigma_{k,k} + \sigma_{i,j} + \eta_{j,i} . \tag{4.21}$$

Moreover, if $N \geq 3$, $t_{[ij]}$ ε $C^1(R)$, and t_{ij} and m_{ij} meet (2.1); then

$$\nabla^2\vartheta_i + e_{ijk}\varphi_{j,k} = -f_i ,$$
$$\nabla^2\eta_i + 2\varphi_i = -c_i . \tag{4.22}$$

Proof. The second part of the theorem follows from Theorem 2.2, and the representation (4.20) of t_{ij} was established in Theorem 4.1. We are left with verifying the representation (4.21) of m_{ij} . By Lemma 4.2 there exist B_{ij} ε $C^{N+2}(R)$ such that

$$m_{ij} = \nabla^4 B_{ij} . \tag{4.23}$$

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*Representation (4.19) gives us a refinement of Gurtin's [5] generalized Beltrami solution of the usual equations of continuum mechanics. If

$$\tau_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} \qquad (\omega_{ji} = \omega_{ij})$$

and $\nabla^2\vartheta_i = -b_i$, then τ_{ij} is a solution of

$$\tau_{ij,i} + b_j = 0, \qquad \tau_{ji} = \tau_{ij} .$$

Setting

$$\Sigma_{ij} = \omega_{ij} - \tfrac{1}{2}\delta_{ij}\omega_{kk} \qquad (\omega_{ij} = \Sigma_{ij} - \delta_{ij}\Sigma_{kk})$$

we obtain Schaefer's [2] solution

$$\tau_{ij} = -\nabla^2\Sigma_{ij} + \Sigma_{im,mj} + \Sigma_{jm,mi} - \delta_{ij}\Sigma_{mn,mn} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} .$$

Again we have the identity

$$\nabla^4 B_{ij} = e_{ila}e_{jmb}e_{lpc}e_{mqd}B_{pq,abcd} - B_{lm,lmij} + \nabla^2(B_{il,lj} + B_{lj,li}). \qquad (4.24)$$

Defining

$$\chi'_{lm} = e_{lpc}e_{mqd}B_{pq,cd} \ ,$$

$$\sigma'_i = B_{il,l} \ , \qquad (4.25)$$

$$\eta_i = \nabla^2 B_{li,l} \ ,$$

we have from Eqs. (4.23) and (4.24) that

$$m_{ij} = e_{imp}e_{jnq}\chi'_{mn,pq} - \sigma'_{k,kij} + \nabla^2\sigma'_{i,j} + \eta_{j,i} \ . \qquad (4.26)$$

Setting

$$\chi_{ij} = \chi'_{ij} + \delta_{ij}\sigma'_{k,k} \ , \qquad (4.27)$$

$$\sigma_i = \nabla^2\sigma'_i$$

completes the proof.

THEOREM 4.3. *Let* $T_{(ij)}$ ε $C^0(R')$, $T_{(ij)}$ ε $C^N(R)$, *and* M_{ijk} ε $C^0(R')$, M_{ijk} ε $C^{N+1}(R)$ *with* $N \geq 2$. *Then there exist* $\omega_{ij} = \omega_{ij}$ ε $C^N(R)$, ϑ_i ε $C^{N-1}(R)$, φ_i , *and* Λ_{ijk} ε $C^{N+1}(R)$, γ_{ij} ε $C^{N+3}(R)$, λ_{ij} ε $C^{N+1}(R)$, ψ_{ij} ε $C^{N-1}(R)$ *such that*

$$T_{ij} = e_{imp}e_{jnq}\omega_{mn,pq} - \delta_{ij}\vartheta_{k,k} + \vartheta_{i,j} + \vartheta_{j,i} + e_{ijk}\varphi_k \qquad (4.28)$$

and

$$M_{ijk} = e_{ilp}e_{jmq}e_{knr}\Lambda_{lmn,pqr} + \gamma_{lm,lmijk} - \nabla^2(\gamma_{il,ljk} + \gamma_{lj,lik})+ \nabla^4\gamma_{ij,k}$$
$$- \lambda_{ik,lij} + \nabla^2\lambda_{ik,j} + \psi_{ik,i} \ . \qquad (4.29)$$

Moreover, if $N \geq 3$, $T_{[ij]}$ ε $C^1(R)$, *and* T_{ij} *and* M_{ijk} *satisfy Eqs. (3.1); then*

$$\nabla^2\vartheta_i + e_{ijk}\varphi_{j,k} = -F_i \ , \qquad (4.30)$$

$$\nabla^2\psi_{[ij]} + e_{ijk}\varphi_k = -C_{[ij]} \ .$$

Proof. The second part of the theorem follows from Theorem 3.1, and the representation (4.28) of T_{ij} was established in Theorem 4.1. The representation (4.29) of M_{ijk} is deduced as follows. By Lemma 4.3 there exist C_{ijk} ε $C^{N+4}(R)$ such that

$$M_{ijk} = \nabla^6 C_{ijk} \ . \qquad (4.31)$$

Also, in view of (4.12), we have the identity

$$\nabla^6 C_{ijk} = -e_{ila}e_{jmb}e_{knc}e_{lpd}e_{mqe}e_{nrf}C_{pqr,abcdef} + C_{lmn,lmnijk}$$
$$- \nabla^2(C_{ilm,lmjk} + C_{ljm,lmik} + C_{lmk,lmij}) + \nabla^4(C_{ljk,li} + C_{ilk,lj} + C_{ijl,lk}). \qquad (4.32)$$

Now define

$$\Lambda_{lmn} = -e_{lpd}e_{mqe}e_{nrf}C_{pqr,def} \ ,$$

$$\gamma_{ij} = C_{ijl,l} \ , \qquad (4.33)$$

$$\lambda_{ij} = \nabla^2 C_{ilj,l} \ ,$$

$$\psi_{ij} = \nabla^4 C_{lij,l} \ ,$$

and the theorem is proved.

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