FIELD DUE TO A CONDUCTING HALF-PLANE AND COPLANAR LINE SOURCE*

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1. Introduction. The problem of constructing the Green's function with harmonic time dependence for the conducting half-plane has been reviewed in an informal report by Levine [1]. In addition, Levine develops his own solution by first expanding in terms of the appropriate eigenfunctions; then by clever manipulation, he converts the series into integrals and obtains an elegant integral expression for the Green's function. An earlier attempt by Harrington [2] to solve this problem by the Wiener-Hopf technique was unsuccessful. Solution by this method hinged on the ability to invert a rather complicated transform for the jump in the normal derivative of the field across the plate. The failure of this approach is not surprising in view of the intricate nature of Levine's result.

For the particular case when the line source lies in an extension of the conducting half-plane, Levine notes that his solution simplifies considerably. Thus, it is natural to inquire if this simplified situation might not be solvable through the Wiener-Hopf technique. Indeed it is, as the following analysis will illustrate.

2. Solution of the Wiener-Hopf problem. We seek the field $E_z = g(x, y; -b, 0)e^{-i\omega t}$ for a perfectly conducting half-plane ($y = 0$, $x > 0$) and a line source at $(-b, 0)$, $b > 0$. Hence

$$\nabla^2 g + k^2 g = -\delta(x, -b)\delta(y, 0), \tag{1}$$

$$g = 0 \quad \text{for} \quad y = 0 \qquad x > 0 \tag{2}$$

where $k = \omega/c$. Further we require that g be such that E_z has the character of an outgoing wave at infinity. A solution of (1) with the proper behavior at infinity is given by

$$g(x, y; -b, 0) = \frac{i}{4} H_0^{(1)}\{k[(x + b)^2 + y^2]^{\frac{1}{2}}\} - \frac{i}{4}\int_0^\infty H_0^{(1)}\{k[(x - s)^2 + y^2]^{\frac{1}{2}}\}f(s)\,ds \tag{3}$$

where

$$f(x) = \left(\frac{\partial g}{\partial x}\right)_{y=0+} - \left(\frac{\partial g}{\partial x}\right)_{y=0-} \qquad x > 0. \tag{4}$$

Imposing the boundary condition (2) yields the integral equation,

$$H_0^{(1)}(k\,|x + b|) = \int_0^\infty H_0^{(1)}(k\,|x - s|)f(s)\,ds \qquad x > 0 \tag{5}$$

An explicit solution for $f(s)$ can be achieved by application of the Wiener-Hopf technique. We begin by first adding a small imaginary part to k, that is

$$\beta = k + i\epsilon, \qquad \epsilon > 0.$$

*Received February 24, 1965.

Then we extend equation (5) to the interval $(-\infty, \infty)$ by adding to the left-hand side a function $\phi_-(x)$ such that

$$\phi_-(x) = \begin{cases} \phi(x) & x < 0 \\ 0 & x > 0 \end{cases}$$

Then taking Fourier transforms yields

$$2i\Phi_-(\alpha) + (2\pi)^{-1/2}(\alpha^2 - \beta^2)^{-1/2}e^{-i\alpha b} = (\alpha^2 - \beta^2)^{-1/2}F_+(\alpha) \qquad -\epsilon < \tau < \epsilon \tag{6}$$

where $\alpha = \sigma + i\tau$ is the transform variable and

$$F_+(\alpha) = (2\pi)^{-1/2} \int_0^\infty f(x)e^{i\alpha x}\, dx \tag{7}$$

$$\Phi_-(\alpha) = (2\pi)^{-1/2} \int_{-\infty}^0 \phi(x)e^{i\alpha x}\, dx \tag{8}$$

By rearrangement of (6) into the equivalent form,

$$2i(\alpha - \beta)^{1/2}\Phi_-(\alpha) + (2\pi)^{-1/2}(\alpha + \beta)^{-1/2}e^{-i\alpha b} \operatorname{Erf}\{[-ib(\alpha + \beta)]^{1/2}\}$$
$$= (\alpha + \beta)^{-1/2}F_+(\alpha) - (2\pi)^{-1/2}(\alpha + \beta)^{-1/2}e^{-i\alpha b} \operatorname{Erfc}\{[-ib(\alpha + \beta)]^{1/2}\}, \qquad -\epsilon < \tau < \epsilon.$$

we note that the left-hand side is analytic for $\tau < \epsilon$ while the right-hand side is analytic for $\tau > -\epsilon$. Since we have equality in the strip, we can use these two expressions to define an entire function. Furthermore, this entire function can be shown to be identically equal to zero by examining the behavior as $|\alpha| \to \infty$. Therefore

$$F_+(\alpha) = (2\pi)^{-1/2}e^{-i\alpha b} \operatorname{Erfc}\{[-ib(\alpha + \beta)]^{1/2}\} \tag{9}$$

Inverting (9) and allowing $\epsilon \to 0$, we obtain

$$f(x) = \frac{b^{1/2}}{\pi} \frac{e^{ik(x+b)}}{x^{1/2}(x + b)} \qquad x > 0, \quad b > 0 \tag{10}$$

Then from (3) follows

$$g(x, y; -b, 0) = \frac{i}{4} H_0^{(1)}\{k[(x + b)^2 + y^2]^{1/2}\}$$
$$- \frac{i}{4} b^{1/2}e^{ikb} \int_0^\infty H_0^{(1)}\{k[(x - s)^2 + y^2]^{1/2}\} \frac{e^{iks}}{s^{1/2}(s + b)}\, ds. \tag{11}$$

3. Concluding remarks. The form of equation (11) is different from the equivalent expression due to Levine, and the means of conversion is not readily apparent. However, if one takes Levine's result and computes the jump in the normal derivative of g across the plate, the result (10) is obtained.

Finally we can express the result for the field E_z as

$$E_z(x, y, t; b) = \frac{i}{4} H_0^{(1)}\left\{\frac{\omega}{c} [(x + b)^2 + y^2]^{1/2}\right\}e^{-i\omega t}$$
$$- \frac{i}{4} b^{1/2}e^{i\omega/c(b-ct)} \int_0^\infty H_0^{(1)}\left\{\frac{\omega}{c} [(x - s)^2 + y^2]^{1/2}\right\} \frac{e^{iks}}{s^{1/2}(s + b)}\, ds,$$

where we note that in the second term (which is the field due to the conducting plate) there is a phase lag in the factor $e^{i\omega/c(b-ct)}$ due to the time required for the disturbance from the line source to reach the leading edge of the plate.

REFERENCES

1. Harold Levine, *The Half-Plane Diffraction Problem for Harmonic Time Dependence,* Tech. Report ONRL-34-60 (1960)
2. R. F. Harrington, *Current Element Near the Edge of a Conducting Half-Plane,* J. of Appl. Physics, **24** 547 (1953)