DIRECTIVITY FOR SCALAR RADIATION*

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How is the energy radiated from a source to be concentrated into a narrow beam? This is the problem of directivity [1]. Directivity is obtained in practice by the use of a reflector or horn. But such a device is not very different from supplementing the original source by the addition of image-sources, and it seems of interest to look into the case where there is no reflector or horn, but only point-sources of radiation, acting independently. If the positions of such point-sources are given, and an assigned frequency, we have at our disposal, in order to obtain a directed beam, only the amplitudes and phases of the source. How should these be chosen in order to maximise directivity in some assigned direction? It is the purpose of this paper to examine that question for scalar radiation. However, it must be realised that, although we have a common *qualitative* understanding of the meaning of the word *directivity* (the radiated energy is concentrated in a narrow beam), when it comes to a *quantitative* definition (so that we can say that one beam is more concentrated than another) one has a certain discretion regarding the suitable definition. Schelkunoff and Friis (op. cit. p. 179) define directivity as the ratio of maximum to average radiation intensity; I define it in a different way.

Consider a set of point-sources at positions P_n ($n = 1, 2, \cdots N$). Let V be the resulting radiation field, satisfying

$$\Delta V - c^{-2}\, \partial^2 V/\partial t^2 = 0. \tag{1}$$

Let us use units for which $c = 1$. The vector representing flux of energy is

$$\mathbf{F} = -(\partial V/\partial t) \text{ grad } V. \tag{2}$$

Then on a large sphere there is a scalar field of density of energy-flux across the sphere, namely F_r, the normal component of $\mathbf{F}$. Our aim is to choose the amplitudes and phases of the sources so that the distribution of F_r is concentrated in the neighbourhood of some point on the sphere.

Restricting the argument to simple harmonic sources with circular frequency ω, we represent the source at P_n by

$$\sigma_n \exp{(i\omega t)} + \sigma_n^* \exp{(-i\omega t)}, \tag{3}$$

where σ_n is a complex constant describing the amplitude and phase of the source. Then the field V at position P and time t, due to this source, is $V_n = W_n + W_n^*$ where

$$W_n(P, t) = -\sigma_n(4\pi PP_n)^{-1} \exp i\omega(t - PP_n). \tag{4}$$

Let O be the origin and $R = OP$; then

$$PP_n^2 = R^2 - 2\mathbf{OP}\cdot\mathbf{OP}_n + OP_n^2, \tag{5}$$

and so, for large R,

$$PP_n = R - \mathbf{OP}_n\cdot\mathbf{I} + O(R^{-1}), \tag{6}$$

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where **I** is the unit vector $\mathbf{OP}/R$. Accordingly (4) gives

$$W_n(P, t) = -\sigma_n(4\pi R)^{-1} \exp i\omega(t - R + \mathbf{OP}_n \cdot \mathbf{I}) + O(R^{-2}), \tag{7}$$

and the total complex field due to all the sources is

$$W(P, t) = \sum_{n=1}^{N} W_n(P, t). \tag{8}$$

From (2) it follows that the time-average of normal flux-density at any point on the sphere R = const. is

$$\langle F_\nu \rangle = -i\omega(W\, \partial W^*/\partial R - W^*\, \partial W/\partial R). \tag{9}$$

By (7) and (8) this is

$$\langle F_\nu \rangle = \omega^2(16\pi^2R^2)^{-1} \sum_{m,n=1}^{N} \sigma_m^* A_{mn}\sigma_n + O(R^{-3}), \tag{10}$$

where A_{mn} is the Hermitean matrix

$$A_{mn} = \exp i\omega \mathbf{P}_{mn} \cdot \mathbf{I}, \tag{11}$$

with

$$\mathbf{P}_{mn} = \mathbf{OP}_n - \mathbf{OP}_m , \tag{12}$$

the position vector of P_n relative to P_m.

Integrating (10) over the infinite sphere, we get for the total flux

$$Q = \int \langle F_\nu \rangle \, dS = \omega^2(4\pi)^{-1} \sum_{m,n=1}^{N} \sigma_m^* Q_{mn}\sigma_n , \tag{13}$$

where Q_{mn} is the real symmetric matrix

$$Q_{mn} = \sin(\omega P_{mn})/(\omega P_{mn}), \tag{14}$$

involving the distances between the sources, made dimensionless by the factor ω.

To discuss directivity, let us define the *moment* of the flux through the infinite sphere as

$$M = \int (1 - \cos\theta) \langle F_\nu \rangle \, dS, \tag{15}$$

where $dS = R^2 \sin\theta \, d\theta \, d\phi$, θ and ϕ being polar angles with $\theta = 0$ in the direction in which concentration is sought. There is of course some arbitrariness in this choice of definition of moment, but the factor $(1 - \cos\theta)$ seems the simplest available; what we need is a factor vanishing for $\theta = 0$ and positive for all other values of θ.

We now define the *directivity* to be

$$D = Q/M. \tag{16}$$

Thus D is large if the beam is highly concentrated near $\theta = 0$.

From (10) and (15) we obtain for the moment

$$M = \omega^2(16\pi^2)^{-1} \sum_{m,n=1}^{N} \sigma_m^* M_{mn}\sigma_n , \tag{17}$$

where M_{mn} is the Hermitean matrix

$$M_{mn} = \int (1 - \cos\theta) \exp(i\omega \mathbf{P}_{mn} \cdot \mathbf{I})\, d\Omega, \tag{18}$$

with $d\Omega = \sin\theta\, d\theta\, d\phi$ and $\mathbf{I} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

The problem of maximising the directivity D is therefore the problem of minimising the quotient of two Hermitean quadratic forms:

$$D^{-1} = \frac{M}{Q} = \frac{\sum_{m,n=1}^{N} \sigma_m^* M_{mn} \sigma_n}{\sum_{m,n=1}^{N} \sigma_m^* Q_{mn} \sigma_n} = \frac{\boldsymbol{\sigma}^\dagger \mathbf{M} \boldsymbol{\sigma}}{\boldsymbol{\sigma}^\dagger \mathbf{Q} \boldsymbol{\sigma}}. \tag{19}$$

The minimum value of D^{-1}, for given positions P_n and a given circular frequency ω, is therefore the smallest root of the equation

$$\det(\mathbf{M} - \lambda \mathbf{Q}) = 0, \tag{20}$$

i.e., the smallest eigenvalue λ given by the equations

$$\mathbf{M}\boldsymbol{\sigma} = \lambda \mathbf{Q} \boldsymbol{\sigma}, \tag{21}$$

and the minimising amplitudes and phases of the sources are given by the corresponding vector $\boldsymbol{\sigma}$.

One could push the matter a little further by expressing the matrix $\mathbf{M}$, as given in (18), in terms of Bessel functions. But that is not important, because formal mathematics cannot really be expected to carry us much further. The minimising of the quotient (19) demands numerical calculation, and it does not seem to matter much how $\mathbf{M}$ is expressed. Given a lattice of sources and a circular frequency ω, we are to try experimentally to reduce the quotient (19) as far as possible by the choice of the complex vector $\boldsymbol{\sigma}$. Having thus obtained an experimental minimum, we should do the whole thing over again for various values of ω in order to minimise the minimum, and when that has been done, we should try to reduce the minimum still further by changing the lattice. In fact, the present paper does not offer a solution of the problem of directivity; its aim is rather to present a systematic approach as a basis for numerical calculations.

Reference

S. A. Schelkunoff and H. T. Friis, *Antennas: Theory and Practice*, Wiley, New York, 1952; in particular Chaps. 6 and 18