A CORRELATION RESULT FOR NONSTATIONARY INPUTS*

BY

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1. Introduction. In [1], K. S. Miller obtained an expression for the mean-square output of a linear time-invariant filter subjected to a nonstationary random input, $y(t) = g(t)x(t)$, where $x(t)$ is a wide-sense stationary random process and $g(t)$ is a deterministic function of the particular form

$$g(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n). \tag{1}$$

The purpose of this note is to generalize Miller's result in two directions; we shall consider arbitrary modulation functions $g(t)$ and also obtain an expression for the general output autocorrelation function

$$\phi_z(t_1, t_2) = E[z(t_1)z(t_2)],$$

where $z(t)$ is the output corresponding to the input $y(t) = g(t)x(t)$. The mean-square output is then easily obtained by taking $t_1 = t_2 = t$. Since Miller's formula is immediate on further specialization of the modulation function, the general analysis affords as a by-product a much simpler proof of the result in [1]. In the concluding section, a reciprocity theorem is proved which shows that the mean-square output of a linear filter is invariant with respect to an interchange of the impulse response $h(t)$ and the modulation function $g(t)$.

In addition to the application suggested by Miller [1], we note that reverberation noise is commonly modeled [2] in the form $g(t)x(t)$, where $g(t)$ represents the time-varying decay characteristic of the reverberation and $x(t)$ is a stationary random process. Thus, nonstationary noises of the more general type considered here are involved in the analysis of sonar detection systems operating in a reverberation-limited environment.

2. Analysis. Let $y(t) = g(t)x(t)$ denote the input to a linear time-invariant system with impulse response $h(t)$, so that the output $z(t)$ is given by

$$z(t) = \int_{-\infty}^{\infty} h(t - \xi)y(\xi)\, d\xi. \tag{2}$$

We assume both $h(t)$ and $g(t)$ are real-valued with $g(t)$ uniformly bounded on $(-\infty, \infty)$ and $h(t)$ in the class $L_1 \cap L_2$ on $(-\infty, \infty)$; that is, $h(t)$ is square integrable and corresponds to a stable filter so that it is also absolutely integrable. The real-valued random process $x(t)$ is assumed wide-sense stationary with square integrable autocorrelation function $\phi_x(\tau) \equiv E[x(t)x(t + \tau)]$ and related power spectral density $S_x(\omega)$.

Using (2) and noting that $E[y(\xi)y(\eta)] = g(\xi)g(\eta)\phi_x(\xi - \eta)$,

$$\phi_z(t_1, t_2) \equiv E[z(t_1)z(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t_1 - \xi)h(t_2 - \eta)g(\xi)g(\eta)\phi_x(\xi - \eta)\, d\xi\, d\eta. \tag{3}$$

*Received August 27, 1965; revised manuscript received October 14, 1965.

We next define a function $B(\omega, t)$ as follows:

$$B(\omega, t) = \int_{-\infty}^{\infty} h(t - \xi)g(\xi)e^{-i\xi\omega}\, d\xi. \tag{4}$$

Then, by Parseval's theorem [3], the ξ-integration in (3) becomes

$$\int_{-\infty}^{\infty} h(t_1 - \xi)g(\xi)\phi_x(\xi - \eta)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega, t_1)S_x(\omega)e^{i\eta\omega}\, d\omega$$

and the remaining η-integration yields

$$\phi_z(t_1, t_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} B(\omega, t_1)B^*(\omega, t_2)S_x(\omega)\, d\omega, \tag{5}$$

where an asterisk superscript denotes complex conjugate. Equation (5) with $B(\omega, t)$ given by (4) is the main result of this section. For $t_1 = t_2 = t$, equations (4) and (5) specialize to

$$E[z^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |B(\omega, t)|^2 S_x(\omega)\, d\omega, \tag{6}$$

which is the desired extension of Miller's result to arbitrary bounded modulation functions $g(t)$.

If, in addition, $g(t) \in L_2$ on $(-\infty, \infty)$, then by Parseval's theorem applied to (4), we have

$$B(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\eta)G(\eta + \omega)e^{i\eta t}\, d\eta, \tag{7}$$

where $H(\omega)$ and $G(\omega)$ are the (L_2) Fourier transforms of $h(t)$ and $g(t)$ respectively. Equation (6) may then be written equivalently in terms of the spectra:

$$E[z^2(t)] = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} H(\eta)G(\eta - \omega)e^{i\eta t}\, d\eta\right|^2 S_x(\omega)\, d\omega. \tag{8}$$

3. Miller's Theorem. To see that (6) contains the previous result [1] as a special case, we consider $g(t)$ to have the particular form

$$g(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n);$$

then from (4),

$$\begin{aligned} B(-\omega, t) &= \frac{1}{2}\sum_{n=1}^{N} a_n \int_{-\infty}^{\infty} h(t - \xi)[e^{i(\omega_n\xi+\phi_n)} + e^{-i(\omega_n\xi+\phi_n)}]e^{i\omega\xi}\, d\xi \\ &= \tfrac{1}{2}e^{i\omega t}[M(\omega) + M^*(-\omega)], \end{aligned} \tag{9}$$

where we have followed Miller in defining

$$M(\omega) \equiv \sum_{n=1}^{N} a_n H(\omega + \omega_n)e^{i(\omega_n t+\phi_n)}.$$

Then, noting that $B(-\omega, t) = B^*(\omega, t)$, substitute (9) in (6) to obtain

$$E[z^2(t)] = \frac{1}{8\pi}\int_{-\infty}^{\infty} |M(\omega) + M^*(-\omega)|^2 S_x(\omega)\, d\omega,$$

which is easily seen to be the equivalent of Miller's formula (equation (4) in [1]).

4. Reciprocity Theorem. In this section we require $g(t)$ [as well as $h(t)$] to belong to $L_1 \cap L_2$ on $(-\infty, \infty)$. An obvious change of variable coupled with the observation that $S_x(\omega)$ is an even function allows us to write (8) in the alternate form:

$$\begin{aligned} E[z^2(t)] &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} H(\eta)G(\eta+\omega)e^{i\eta t}\,d\eta\right|^2 S_x(\omega)\,d\omega \\ &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} H(\xi-\omega)G(\xi)e^{i(\xi-\omega)t}\,d\xi\right|^2 S_x(\omega)\,d\omega. \end{aligned} \tag{10}$$

Rearranging the latter expression and changing the dummy variable back to η, we have

$$E[z^2(t)] = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} G(\eta)H(\eta-\omega)e^{i\eta t}\,d\eta\right|^2 S_x(\omega)\,d\omega. \tag{11}$$

Comparison of (8) and (11) proves the following reciprocity result:

Let $x(t)$ be a wide-sense stationary random process with square integrable power spectral density $S_x(\omega)$. If $h(t)$ and $g(t)$ are two given real-valued functions in $L_1 \cap L_2$ on $(-\infty, \infty)$, then the mean-square output of a time-invariant linear filter with impulse response $h(t)$ and input $y(t) = g(t)x(t)$ is exactly the same as the mean-square output of a linear system with impulse response $g(t)$ and input $y(t) = h(t)x(t)$.

Thus, the mean-square output of a linear system is invariant with respect to an interchange in roles of the impulse response $h(t)$ and the modulation function $g(t)$.

Lastly, we observe that the time dependent spectral density $W(t, \omega)$ defined by

$$W(t, \omega) \equiv \int_{-\infty}^{\infty} \phi_z(t, t+\tau)e^{-i\omega\tau}\,d\tau$$

can be calculated from (5) to give the following result:

$$W(t, \omega) = \frac{e^{i\omega t}H(\omega)}{2\pi}\int_{-\infty}^{\infty} B(\alpha, t)G(\omega-\alpha)S_x(\alpha)\,d\alpha. \tag{12}$$

This formula generalizes equation (16) of Miller's paper [1] and can easily be shown to reduce to that result when $g(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \phi_n)$. For the special case,

$$G(\omega) = \pi\sum_{n=1}^{N} a_n[e^{i\phi_n}\delta(\omega-\omega_n) + e^{-i\phi_n}\delta(\omega+\omega_n)],$$

where $\delta(\omega)$ is the delta functional and $B(\alpha, t)$ in (12) is given by $(e^{-i\alpha t}/2)[M(-\alpha)+M^*(\alpha)]$ from equation (9).

References

1. K. S. Miller, *A note on input-output spectral densities,* Q. Appl. Math. **21**, 249 (1963)
2. V. V. Ol'shevskii, *Probability distribution of sea reverberation levels,* Soviet Physics-Acoustics, **9**, 378 (1964)
3. R. R. Goldberg, *Fourier Transforms,* Cambridge University Press, p. 48 (1961)