SERIES REPRESENTATION OF CONTINUOUS FUNCTIONS*

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1. Introduction. A representation series is derived here for a class of continuous functions Φ on the t-interval $I' = [0, 1]$. If $\varphi(t) \in \Phi$ on a subset σ of I' then:

$$\varphi(t) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \varphi_j^*(k^{i+1}\tau_j) \right], \qquad t \in \sigma \tag{1.1}$$

where τ_j is defined by the continuous function $\tau_j = f_j(t) \in [0, 1]$ for $t \in I'$, and $\sum_{j=1}^{m} \varphi_j^*(\tau_j) = \varphi(t)$. This series representation converges *pointwise* to $\varphi(t)$ for $t \in \sigma$. The parameter k ranges over values $0 \leq k(t) < k < 1$, where $k(t)$ is determined by the local properties of $\varphi(t)$ at $t \in \sigma$.

For the class Φ' of functions $\varphi(t)$ with the property that $\sigma = I'$, this representation is shown to be unique within Φ'. That is, there exists no other set of functions $\psi_j(\tau_j) \in \Phi'$ such that:

$$\varphi(t) = \sum_{j=1}^{m'} \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v}(k) \psi_j(k^{i+1}\tau_j)$$

other than $\sum_{j=1}^{m'} \psi_j(\tau_j) = \sum_{j=1}^{m} \varphi_j^*(\tau_j) = \varphi(t)$, $\tau_j \in I'$. The representation series (1.1) and the uniqueness properties of the class Φ' are the contents of Theorem 1.

It should be noticed that this representation series for continuous functions is more similar to the Taylor series representation for analytic functions than to the expansion of continuous functions using the concept of convergence in the mean. Clearly the convergence concept of this representation series and the Taylor series are the same, which result in the fact that both series depend strongly on the function itself. There are several other analogies between the two series. One point of analogy is between the radius of convergence concept and the value $k(t)$.

Theorem 2 in section 3 deals with the continuation of a function $\varphi \in \Phi'$. If $\varphi \in \Phi'$, Theorem 2 states that it is sufficient to define each $\varphi_j^*(\tau_j)$ on a subinterval $\tau_j \in [0, \delta]$, $0 < \delta < 1$ in order to completely define φ in I'. In some sense, to be discussed in section 3, there is an analogy between the continuation of $\varphi \in \Phi'$ and analytic continuation.

Section 4 exhibits a class of functions $\varphi \in \Phi'$, which is not analytic in I'. The class of continuous functions $\pi(t)$, defined by a finite set of joined polynomial segments is shown to belong to Φ'.

Since series (1.1) is a discrete point representation of a continuous function, one is immediately aware of the possibility of its application to the fields of numerical analysis and data processing. The representation series in truncated form, is an extrapolation formula over a definite set of grid points. The grid points, $k\tau_j$, $k^2\tau_j$, $k^3\tau_j$, $\cdots$, are the points at which the known function values are taken.** There is a class of numerical

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**These grid points are self generating in the sense that if one choses a k, and extrapolates to τ_j by using the values $k^{i+1}\tau_j$, $i = 0, \cdots, n$ one can then find the point τ_j/k by using the same data.

problems for which this truncated form of the series might be helpful. A more fundamental problem of numerical analysis for which this series may be used concerns the general approach to numerical analysis. For a large class of problems, one must almost immediately assume that the solution is analytic in order to construct a numerical model of the mathematical solution. The above series might furnish some greater latitude in this area, since it covers a more general class of functions.

There are some major restrictions, however, that are an essential property of this series. While the series is absolutely convergent with respect to the v summation, it is not absolutely convergent if we consider it as a series of the individual terms in the brackets. This fact does not allow one to rearrange the series in the form $\sum \beta_\mu \varphi^*(k^\mu \tau)$, and effectively removes the procedure of comparison of coefficients as an effective tool which may be applied to the series.

2. The Series Representation. It is convenient to consider a function $x(t)$ analytic on the closed interval $[-1, 1]$. Since one can always consider a closed subinterval of any interval of analyticity this presents no problem. Let $x(t)$ be analytic in some interval $(-a^2, a^2)$ where $a^2 > 1$, we shall now prove $x(t) \varepsilon \Phi'$.

LEMMA 1. If $x(t)$ is an analytic function in the t-interval, $I = [-1, 1]$, and the parameter k lies in the region $0 < k < 1$, then for any $t \varepsilon I$ the following relation holds, where the series on the right hand side is absolutely convergent:

$$x(t) = 2x(kt) - x(k^2t) + \sum_{v=1}^{\infty} \int_{kt}^{t} [(2 - k)x^{(v)}(kt) - x^{(v)}(k^2t)](dt)^v \tag{2.1}$$

where $x^{(v)}(kt) = (d^v/d\eta^v)x(\eta) \mid_{\eta = kt}$ and $\int_{kt}^{t} (dt)^v$ denotes the vth fold integral:

$$\underbrace{\int_{kt}^{t} \cdots \int_{kt}^{t}}_{v \text{ times}} \underbrace{dt \cdots dt}_{v \text{ times}}.$$

Proof: Since $t \varepsilon I$, and $0 < k < 1$, $x(t)$ has a Taylor series expansion about $x(kt)$ and $x(kt)$ has a Taylor series expansion about $x(k^2t)$, i.e.,

$$x(t) = \sum_{i=0}^{\infty} \frac{(1 - k)^i t^i}{i!} x^{(i)}(kt)$$

and

$$x(kt) = \sum_{i=0}^{\infty} \frac{(1 - k)^i (kt)^i}{i!} x^{(i)}(k^2t).$$

Let us integrate the series:

$$\sum_{v=1}^{\infty} \int_{kt}^{t} [(2 - k)x^{(v)}(kt) - x^{(v)}(k^2t)](dt)^v$$
$$= \sum_{v=1}^{\infty} \int_{kt}^{t} \left[(1 - k)x^{(v)}(kt) + \sum_{i=1}^{\infty} \frac{(1 - k)^i (kt)^i}{i!} x^{(v+i)}(k^2t)\right](dt)^v \tag{2.2}$$

Integrating the first term of each integral in the right hand side of equation (2.2) by parts we get:

$$\sum_{v=1}^{\infty} \int_{kt}^{t} [(2 - k)x^{(v)}(kt) - x^{(v)}(k^2t)](dt)^v$$

$$= \sum_{\nu=1}^{\infty} \int_{kt}^{t} \Big[(1-k)(tx^{(\nu)}(kt) - ktx^{(\nu)}(k^2t)) - \int_{kt}^{t} (1-k)ktx^{(\nu+1)}(kt)\, dt$$
$$+ \int_{kt}^{t} \sum_{i=1}^{\infty} \frac{(1-k)^i (kt)^i}{i!} x^{(\nu+i)}(k^2t)\, dt \Big] (dt)^{\nu-1} \tag{2.3}$$
$$= (1-k)tx^{(1)}(kt) - (1-k)ktx^{(1)}(k^2t) + \sum_{\nu=1}^{\infty} \int_{kt}^{t} \Big[(1-k)^2 tx^{(\nu+1)}(kt)$$
$$+ \sum_{i=2}^{\infty} \frac{(1-k)^i (kt)^i}{i!} x^{(\nu+i)}(k^2t) \Big] (dt)^{\nu}.$$

We now proceed by induction. Suppose that we repeat this procedure m times and at the mth time get:

$$\sum_{\nu=1}^{\infty} \int_{kt}^{t} [(2-k)x^{(\nu)}(kt) - x^{(\nu)}(k^2t)](dt)^{\nu}$$
$$= \sum_{i=1}^{m} \frac{(1-k)^i t^i}{i!} x^{(i)}(kt) - \sum_{i=1}^{m} \frac{(1-k)^i (kt)^i}{i!} x^{(i)}(k^2t)$$
$$+ \sum_{\nu=1}^{\infty} \int_{kt}^{t} \Big[\frac{(1-k)^{m+1}}{m!} t^m x^{(\nu+m)}(kt) + \sum_{i=m+1}^{\infty} \frac{(1-k)^i (kt)^i}{i!} x^{(\nu+i)}(k^2t) \Big] (dt)^{\nu} \tag{2.4}$$

Integrating the first term of each integral by parts, we get:

$$\sum_{\nu=1}^{\infty} \int_{kt}^{t} [(2-k)x^{(\nu)}(kt) - x^{(\nu)}(k^2t)](dt)^{\nu}$$
$$= \sum_{i=1}^{m} \frac{(1-k)^i t^i}{i!} x^{(i)}(kt) - \sum_{i=1}^{m} \frac{(1-k)^i (kt)^i}{i!} x^{(i)}(k^2t)$$
$$+ \sum_{\nu=1}^{\infty} \int_{kt}^{t} \Big[\frac{(1-k)^{m+1}}{(m+1)!} \{t^{m+1} x^{(\nu+m)}(kt) - (kt)^{m+1} x^{(\nu+m)}(k^2t)\}$$
$$- \frac{(1-k)^{m+1}}{(m+1)!} \int_{kt}^{t} kt^{m+1} x^{(\nu+m+1)}(kt)\, dt + \int_{kt}^{t} \sum_{i=m+1}^{\infty} \frac{(1-k)^i}{i!} (kt)^i x^{(\nu+i)}(k^2t)\, dt \Big] (dt)^{\nu-1}$$
$$= \sum_{i=1}^{m+1} \frac{(1-k)^i}{i!} t^i x^{(i)}(kt) - \sum_{i=1}^{m+1} \frac{(1-k)^i (kt)^i}{i!} x^{(i)}(k^2t)$$
$$+ \sum_{\nu=1}^{\infty} \int_{kt}^{t} \Big[\frac{(1-k)^{m+2}}{(m+1)!} t^{m+1} x^{(\nu+m+1)}(kt) + \sum_{i=m+2}^{\infty} \frac{(1-k)^i}{i!} (kt)^i x^{(\nu+i)}(k^2t) \Big] (dt)^{\nu}$$

which is exactly the form we seek. Since equation (2.3) shows that this form holds for $m = 1$, we have proved by induction that:

$$\sum_{\nu=1}^{\infty} \int_{kt}^{t} [(2-k)x^{(\nu)}(kt) - x^{(\nu)}(k^2t)](dt)^{\nu}$$
$$= \sum_{i=1}^{\infty} \frac{(1-k)^i}{i!} t^i x^{(i)}(kt) - \sum_{i=1}^{\infty} \frac{(1-k)^i}{i!} (kt)^i x^{(i)}(k^2t) = x(t) - 2x(kt) + x(k^2t)$$

This completes the proof of Lemma 1. It is clear that equation (2.1) is nothing but a rearrangement of the Taylor series of $x(t)$ and it therefore converges absolutely for $t \in I$. Equation (2.4) also shows that a truncation

$$2x(kt) - x(k^2t) + \sum_{\nu=1}^{m} \int_{kt}^{t} [(2-k)x^{(\nu)}(kt) - x^{(\nu)}(k^2t)](dt)^{\nu},$$

results in an error which is of the same order in t as the truncation, $\sum_{v=0}^{m} (t^v/v!)(1 - k)^v x^{(v)}(kt)$, of the Taylor series of $x(t)$. It should be noted that if $x(t)$ is analytic for $t \in [0, a]$, $0 < a \leq 1$ then $\frac{1}{2} < k < 1$.

Let us now introduce the following notation:
let

$$\overset{v}{\underset{i=1}{S}} = \sum_{i=1}^{v} (-1)^i \sum_{j_i=i}^{v} \sum_{j_{i-1}=i-1}^{j_i-1} \sum_{j_{i-2}=i-2}^{j_{i-1}-1} \cdots \sum_{j_1=1}^{j_2-1}$$

and let

$$\sum_{\alpha=1}^{i} j_\alpha = \sum^i j_\alpha ,$$

Lemma 2. The following relation holds for $t \in I$, $v \geq 1$:

$$\int_{kt}^{t} x^{(v)}(kt)(dt)^v = \frac{x(kt)}{k^v} + \overset{v}{\underset{i=1}{S}} \left[\frac{x(k^{i+1}t)}{k^{(i+1)v - \sum^i j_\alpha}}\right]. \tag{2.5}$$

Proof: We proceed by induction. It is clear that Lemma 2 holds for $v = 1$, i.e.,

$$\int_{kt}^{t} x^{(1)}(kt)\, dt = \frac{x(kt)}{k} - \frac{x(k^2t)}{k}.$$

Suppose that equation (2.5) holds for v, we will prove it holds for $v + 1$.

$$\int_{kt}^{t} x^{(v+1)}(kt)(dt)^{v+1} = \int_{kt}^{t} \left\{\frac{x^{(1)}(kt)}{k^v} + \overset{v}{\underset{i=1}{S}} \left[\frac{x^{(1)}(k^{i+1}t)}{k^{(i+1)v - \sum^i j_\alpha}}\right]\right\} dt$$

$$= \frac{x(kt)}{k^{v+1}} \quad \frac{x(k^2t)}{k^{v+1}} + \overset{v}{\underset{i=1}{S}} \left[\frac{1}{k^{(i+1)v - \sum^i j_\alpha}} \left(\frac{x(k^{i+1}t)}{k^{i+1}} - \frac{x(k^{i+2}t)}{k^{i+1}}\right)\right]$$

$$= \frac{x(kt)}{k^{v+1}} - \sum_{j_1=1}^{v+1} \frac{x(k^2t)}{k^{2(v+1) - j_1}} + \overset{v}{\underset{i=2}{S}} \left[\frac{x(k^{i+1}t)}{k^{(i+1)(v+1) - \sum^i j_\alpha}}\right]$$

$$+ \sum_{i=2}^{v} \sum_{j_i=v+1}^{v+1} \sum_{j_{i-1}=i-1}^{v} \cdots \sum_{j_1=1}^{j_2-1} \left[\frac{(-1)^i x(k^{i+1}t)}{k^{(i+1)(v+1) - \sum^i j_\alpha}}\right]$$

$$+ (-1)^{v+1} \sum_{j_{v+1}=v+1}^{v+1} \sum_{j_v=v}^{v} \cdots \sum_{j_1=1}^{1} \frac{x(k^{v+2}t)}{k^{(v+2)(v+1) - \sum^{v+1} j_\alpha}}$$

$$= \frac{x(kt)}{k^{v+1}} + \overset{v+1}{\underset{i=1}{S}} \left[\frac{x(k^{i+1}t)}{k^{(i+1)(v+1) - \sum^i j_\alpha}}\right] \tag{2.6}$$

which completes the proof of Lemma 2.

At this point it is convenient to introduce the notation

$$\beta_{0,v}(k) = \frac{1}{k^v}$$

$$\beta_{i,v}(k) = (-1)^i \sum_{j_i=i}^{v} \sum_{j_{i-1}=i-1}^{j_i-1} \cdots \sum_{j_1=1}^{j_2-1} \frac{1}{k^{(i+1)v-\sum_i^i j_\alpha}}, \qquad i = 1, \cdots, v.$$

$$\beta_{i,v} = 0 \quad \text{for} \quad i > v$$

We also see that since:

$$\int_{kt}^{t} x^{(v)}(k^2t)(dt)^v = \frac{1}{k^v} \int_{kt}^{t} x^{(v)}(k^2t)(dkt)^v,$$

by Lemma 2 we get:

$$\int_{kt}^{t} x^{(v)}(k^2t)(dt)^v = \frac{1}{k^v} \left\{ \frac{x(k^2t)}{k^v} + \mathop{S}_{i=1}^{v} \left[\frac{x(k^{i+2}t)}{k^{[(i+1)v-\sum_i^i j_\alpha]}} \right] \right\}$$

Let us define

$$\gamma_{0,v}(k) = \frac{1}{k^v} \beta_{0,v}(k)$$

$$\gamma_{i,v}(k) = \frac{1}{k^v} \beta_{i,v}(k).$$

Using Lemma 1, we can than write*:

$$x(t) = \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) x(k^{i+1}t) \right], \qquad t \in I$$

where:

$$\alpha_{0,0} = 2$$

$$\alpha_{1,0} = -1$$

$$\alpha_{0,v} = (2-k)\beta_{0,v}(k) \qquad v \neq 0$$

$$\alpha_{i,v} = (2-k)\beta_{i,v}(k) - \gamma_{i-1,v}(k), \qquad i = 1, \cdots, v$$

$$\alpha_{v+1,v} = -\gamma_{v,v}(k)$$

It should be noticed that Lemmas 1 and 2 prove that our representation series is valid for analytic functions in their domain of analyticity, that is, for an analytic function x, we have $j = 1$, $\tau_1 = t$ and $\varphi_1^* = x$.

From equation (2.6) we see that

$$\sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \right] = 1,$$

but we also see that while the series is absolutely convergent over v, it is not absolutely convergent when we consider it a series over the individual terms in the brackets.

*It is assumed that there exists no finite integer $m \geq 0$ such that $x^{(m)}(t) \equiv 0$ on I. If such an m exists, it is clear that:

$$x(t) = \sum_{v=0}^{m-1} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) x(k^{i+1}t) \right]$$

i.e., in this case we prescribe $\alpha_{i,v} = 0$ for $v \geq m$.

Let us define the polynomial $P_{ink t_1}(\tau_j)$ to be that nth order polynomial such that for a fixed k, $t_1 \varepsilon I'$ and for $i = 1, \cdots, n+1$, $P_{ink t_1}(k^i \tau_{1j}) = \varphi_j^*(k^i \tau_{1j})$ where $\tau_j = f_j(t) \varepsilon I'$ for $t \varepsilon I'$ and $\tau_{1j} = f_j(t_1)$. We now define the class of functions Φ.

Definition 1. $\varphi(t) \varepsilon \Phi$ at $t_1 \varepsilon I'$, if there exist a finite set of continuous transformations $\tau_j = f_j(t)$; $\tau_j \varepsilon I'$ for $t \varepsilon I'$ and a finite set of continuous functions $\varphi_j^*(\tau_j)$; $j = 1, \cdots, m$; such that $\sum_{j=1}^m \varphi_j^*(\tau_j) = \varphi(t)$, where* each $\varphi_j^*(\tau_j)$ has $\{P_{ink t_1}(\tau_j)\}$ as a polynomial approximating sequence for $\tau_j \varepsilon [0, f_j(t_1)]$ and all k in $k(t_1) < k < 1$.

Lemma 3. If $\psi(\tau)$ is continuous in $[0, \tau_1]$ and the series $\sum_{v=0}^\infty \sum_{i=0}^{v+1} \alpha_{i,v}\psi(k^{i+1}\tau_1)$ converges for $k(\tau_1) < k < 1$, $\tau_1 \varepsilon I'$, then

$$\psi(\tau_1) = \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v}\psi(k^{i+1}\tau_1) \quad \text{for} \quad k(\tau_1) < k < 1.$$

Proof: If $\psi(\tau)$ is continuous in $[0, \tau_1]$ then it has some polynomial approximating sequence $\{P_n(\tau)\}$ in $[0, \tau_1]$ where n denotes the order of the polynomial, then:

$$\epsilon_n \geq |\psi(\tau_1) - P_n(\tau_1)| = \left|\psi(\tau_1) - \sum_{v=0}^{n} \sum_{i=0}^{v+1} \alpha_{i,v}P_n(k^{i+1}\tau_1)\right|$$
$$= \left|\psi(\tau_1) - \sum_{v=0}^{n} \sum_{i=0}^{v+1} \alpha_{i,v}\psi(k^{i+1}\tau_1) - \sum_{v=0}^{n} \sum_{i=0}^{v+1} \alpha_{i,v}\epsilon_n'(k^{i+1}\tau_1)\right|$$

where

$$P_n(\tau) - \psi(\tau) = \epsilon_n'(\tau) \qquad \tau \varepsilon [0, \tau_1].$$

Since $\lim_{n\to\infty} \epsilon_n = 0$ and since $\lim_{n\to\infty} \sum_{v=0}^n \sum_{i=0}^{v+1} \alpha_{i,v}\psi(k^{i+1}\tau_1)$ converges then $\lim_{n\to\infty} \sum_{v=0}^n \sum_{i=0}^{v+1} \alpha_{i,v}\epsilon_n'(k^{i+1}\tau_1)$ converges and since $\lim_{n\to\infty} \epsilon_n'(\tau) = 0$, we have:

$$\lim_{n\to\infty} \sum_{v=0}^{n} \sum_{i=0}^{v+1} \alpha_{i,v}\epsilon_n'(k^{i+1}\tau_1) = 0.$$

Therefore,

$$\psi(\tau_1) = \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v}\psi(k^{i+1}\tau_1).$$

Theorem 1. Let $\varphi(t) \varepsilon \Phi$ at $t_1 \varepsilon I'$. Then for any k in $0 \leq k(t_1) < k < 1$, $\varphi(t)$ has the following representation:

$$\varphi(t_1) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k)\varphi_j^*(k^{i+1}\tau_{1j})\right].$$

If in addition $\varphi \varepsilon \Phi'$, then this representation is unique in Φ'. That is, there exists no other set of functions $\psi_j(\tau_j) \varepsilon \Phi'$ such that:

$$\varphi(t_1) = \sum_{j=1}^{m'} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k)\psi_j(k^{i+1}\tau_{1j})\right]$$

other than

$$\sum_{j=1}^{m'} \psi_j(\tau_j) = \sum_{j=1}^{m} \varphi_j^*(\tau_j) = \varphi(t), \qquad t \varepsilon I'.$$

*The stress upon the sum of $\varphi_j^*(\tau_j)$ in the definition is designed so that if $\varphi_1(t), \varphi_2(t) \epsilon \Phi$ at t, then the sum $\varphi_1(t) + \varphi_2(t) \epsilon \Phi$ at t.

Proof: Since $\varphi \in \Phi$ at $t_1 \in I'$, for any k in $k(t_1) < k < 1$, there exists the set of $\varphi_j^*(\tau_j)$, $\sum_{j=1}^m \varphi_j^*(\tau_j) = \varphi(t)$, with the polynomial approximating sequence $\{P_{jnkt_1}(\tau_j)\}$ in $[0, f_j(t_1)]$. Consider the sum:

$$\beta_n(t_1) = \sum_{j=1}^{m} \sum_{v=0}^{n} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \varphi_j^*(k^{i+1} \tau_{1j}) \right].$$

If we use the series representation for $P_{jnkt_1}(\tau_j)$(see footnote (on p. 207)) we get:

$$\left| \beta_n(t_1) - \sum_{j=1}^{n} P_{jnkt_1}(\tau_{1j}) \right| = \left| \sum_{j=1}^{m} \sum_{v=0}^{n} \sum_{i=0}^{v+1} \alpha_{i,v}(k)(\varphi_j^*(k^{i+1}\tau_{1j}) - P_{jnkt_1}(k^{i+1}\tau_{1j})) \right| = 0$$

Then:

$$\lim_{n\to\infty} \left| \beta_n(t_1) - \sum_{j=1}^{m} P_{jnkt_1}(\tau_{1j}) \right| = 0$$

and therefore,*

$$\lim_{n\to\infty} \sum_{j=1}^{m} P_{jnkt_1}(\tau_{1j}) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \varphi_j^*(k^{i+1}\tau_{1j}) \right].$$

But since the $P_{jnkt_1}(\tau_j)$ are approximating polynomials to the $\varphi_j^*(\tau_j)$ in $[0, f_j(t_1)]$ we have that:

$$\varphi(t_1) = \sum_{j=1}^{m} \varphi_j^*(\tau_{1j}) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \varphi_j^*(k^{i+1}\tau_{1j}) \right]$$

This completes the representation part of the proof. We now prove uniqueness.

Let $\varphi \in \Phi'$. From Lemma 3 it is clear that if $\varphi \in \Phi'$ then all the $\varphi_j^* \in \Phi'$. Let $\psi_j \in \Phi'$, $j = 1, \cdots, m'$ and suppose that both:

$$\varphi(t) = \sum_{j=1}^{m} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v} \varphi_j^*(k^{i+1}\tau_j) \right], \quad \text{for} \quad k_\varphi(t) < k < 1,$$

and

$$\varphi(t) = \sum_{j=1}^{m'} \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v} \psi_j(k^{i+1}\tau_j) \right], \quad \text{for} \quad k_\psi(t) < k < 1,$$

hold for $t \in I'$. We want to show $\sum_{j=1}^m \varphi_j^*(\tau_j) = \sum_{j=1}^{m'} \psi_j(\tau_j)$ on I'. The proof is very simple, i.e., if we take $k'(t) = \max(k_\psi(t), k_\varphi(t))$, by Lemma 3,

$$\psi_j(\tau_j) = \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \psi_j(k^{i+1}\tau_j) \right]$$

and

$$\varphi_j^*(\tau_j) = \sum_{v=0}^{\infty} \left[\sum_{i=0}^{v+1} \alpha_{i,v}(k) \varphi_j^*(k^{i+1}\tau_j) \right] \quad \text{at each} \quad t \in I', \qquad k'(t) < k < 1;$$

$$\therefore \varphi(t) = \sum_{j=1}^{m'} \psi_j(\tau_j) = \sum_{j=1}^{m} \varphi_j^*(\tau_j) \quad \text{for} \quad t \in I'.$$

This completes the proof of theorem 1.

*Since $\{P_{jnkt_1}\}$ is a polynomial approximating sequence there is no problem concerning convergence of the series, one could therefore also use Lemma 3 to complete the proof.

In the sense that a continuous function of m variables may be approximated by polynomials in m variables, one can define the approximation of the functions of m variables, $\varphi_j^*(\tau_j)$ in terms of the polynomials $P_{jnk\tau_j}(\tau_j)$, where τ_j in the vector $\tau_j = (f_{1j}(\xi_1, \cdots, \xi_m), \cdots, f_{mj}(\xi_1, \cdots, \xi_m))$. It can then be seen that we can extend definition 1 and concepts in Lemma 3 and theorem 1 to continuous function of several variables.

3. The continuation of functions $\varphi \in \Phi'$. Let us introduce this section with a discussion. The main question that we ask here is as follows: Suppose $\varphi(t) \in \Phi'$, is it possible that by using the uniqueness properties of φ on I', one need only define the φ_j^* on a proper τ_j-subinterval I'' of I', $I'' = [0, \delta]$, $1 > \delta > 0$, to completely define φ on I'?

If this conjecture were true it would provide a mechanism for the unique continuation of continuous function $\varphi \in \Phi'$. One cannot continue φ from I'' to I' simply by taking the values of φ_j^* on I'' and by using equation (1.1) to generate $\varphi(t)$ for $t \in I' - I''$. At first one must establish the existence of a $k(t)$ for each $t \in I' - I''$, that could somehow be associated with φ. This implies that to continue φ we must have, to start with, a function $\varphi' \in \Phi$ for all $t \in I''$ such that $\varphi' = \varphi$ on I''. Since continuous functions are unique in Φ', this means that $\varphi' = \varphi$ on I'.

The role that $k(t)$ plays in this process is analogous to the role that the radius of convergence plays in analytic continuation. At each point t, $k(t)$ determines the region of k, i.e., $0 \leq k(t) < k < 1$ on which equation (1.1) is valid. $k(t)$ therefore determines the upper bound for the distance from t of the first point kt, at which the right hand side of equation (1.1) can be evaluated. $k(t)$, $t \in I' - I''$, therefore, determines how far we can continue φ in one step. The process of continuation may possibly be repeatedly performed in a sequence of steps analogous to a sequence of steps of analytic continuation.

Clearly every point to which we continue $\varphi(t)$ must by definition be a point at which $\varphi \in \Phi'$. Therefore, using equation (1.1) a continuous function may only be continued as a continuous function in Φ'. Therefore, continuation implies the existence of a continuous function in Φ' which agrees with the original function in I'. This fact is analogous to the fact that for analytic continuation the analytic function must agree in their region of common definition.

THEOREM 2. Let $\varphi \in \Phi'$. It is sufficient to define $\varphi_j^*(\tau_j)$ $j = 1, \cdots, m$ on the τ_j-interval $[0, \delta]$ for any δ, $0 < \delta < 1$, in order to completely define $\varphi(t)$ in I'.

Proof: All we have to prove is that by defining each $\varphi_j^*(\tau_j)$ on $[0, \delta]$ we completely define $\varphi_j^*(\tau_j)$ on all $\tau_j = f_j(t)$, $t \in I'$.

At each $t \in I'$ there is a $k(t)$, $0 \leq k(t) < 1$. We claim that there exists a number γ, $0 < \gamma < 1$, such that for all but possibly a finite number of the $t \in I'$, $k(t) < \gamma$. Otherwise, there would exist a sequence of $t_i \in I'$ such that $\lim_{i\to\infty} k(t_i) = 1$. But since I' is closed this means that there exist a $T \in I'$, $T = \lim_{i\to\infty} (t_i)$, such that $k(T) = 1$, i.e., φ does not belong to Φ' at T. This is contrary to our assumptions. Since γ holds for all but a finite number of points and because $\varphi \in \Phi'$, then exists a γ', $\gamma \leq \gamma' < 1$, such that for all $t \in I'$, $k(t) < \gamma'$.

Let $\kappa = 1 + \gamma'$. We then have that in one step we can continue each of the $\varphi_j^*(\tau_j)$ from the τ_j-interval $[0, \delta]$ to the τ_j-interval $[0, \min(1, \kappa\delta)]$. After l steps we can continue $\varphi_j^*(\tau_j)$ to the τ_j—interval $[0, \min(1, \kappa^l\delta)]$. Since $\kappa > 1$ there exists an integer ω such that $\kappa^\omega \geq 1/\delta$. This means that after ω steps $\varphi_j^*(\tau_j)$ would be uniquely defined for any value of $\tau_j \in [0, 1]$ for $j = 1, \cdots, m$. This completes the proof of Theorem 2. Theorem 2 could be extended to higher dimensional space as was Theorem 1.

4. A class of functions in Φ. If $\varphi(t)$ is an analytic function Lemmas 1 and 2 show that $j = 1$, $\tau_1 = t$, and $\varphi_1^* = \varphi$. In this section we shall explore a class of continuous functions in Φ which are not analytic. We shall encounter two types of such functions. One type has the property $\tau = \tau_j = \tau_{j+1}$, $j = 1, \cdots, m-1$, i.e., $\varphi(t) = \sum_{j=1}^m \varphi_j^*(\tau) = \psi(\tau)$. Our series representation is very useful in this case since except for the development with respect to τ rather than t, i.e.,

$$\varphi(t) = \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v} \psi(k^{i+1}\tau),$$

the series representation is formally identical with that of analytic functions. For the other type of functions, in which the τ_j are different, the series representation is less convenient, i.e., in the form of equation (1.1).

A subclass of functions of the second type are the functions $\pi(t)$, defined by dividing the interval [0, 1] into a finite set of l adjoining line segments and in each of these segments defining $\pi(t)$ to be a polynomial $P_i(t)$, $i = 1, \cdots, l$. Of course, since $\pi(t)$ is continuous, $P_i(t)$ are such that $P_i = P_{i+1}$ at their point of common definition. The methods in this section are related to the methods used in the Lebesgue proof of the Weirstrass approximation theorem.

THEROEM 3. Let $\varphi(\tau) = \tau^l \sum_{\mu=0}^{\infty} A_\mu \tau^\mu$, l a non-negative integer, then $\varphi \in \Phi$(for $0 < k < 1$) at any value of τ, $\tau \in I$, at which the series converges.

Proof: Let us first define the polynomials of order s:

$$\sigma_s(\tau) = \tau^l \sum_{\mu=0}^{s-l} A_\mu \tau^\mu.$$

The polynomial sequence $\{\sigma_s(\tau)\}$ is an approximating sequence to $\varphi(\tau)$. Since $\sigma_s(\tau)$ is an analytic function $|\tau| \leq 1$ we can write for $0 < k < 1$;

$$\epsilon_s \geq |\varphi(\tau) - \sigma_s(\tau)| = \left| \varphi(\tau) - \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}(\sigma_s(k^{i+1}\tau) - \varphi(k^{i+1}\tau) + \varphi(k^{i+1}\tau)) \right|$$
$$\geq \left| \varphi(\tau) - \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}\varphi(k^{i+1}\tau) \right| - \left| \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}(\sigma_s(k^{i+1}\tau) - \varphi(k^{i+1}\tau)) \right| \cdot$$

Since $\lim_{s\to\infty} \epsilon_s = 0$ in order to prove that $\varphi(\tau) \in \Phi$ at some $\tau \in I$ we want,

$$\lim_{s\to\infty} \left| \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}(\sigma_s(k^{i+1}\tau) - \varphi(k^{i+1}\tau)) \right| = 0. \tag{4.2}$$

Using the series representation of $\varphi(\tau)$ we get:

$$\left| \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}(\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)) \right| = \left| \sum_{v=0}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}(k^{i+1}\tau)^l \sum_{\mu=s-l+1}^{\infty} A_\mu (k^{i+1}\tau)^\mu \right| \tag{4.3}$$

and since for the $v = 0$ terms we have:

$$\lim_{s\to\infty} |\varphi(k\tau) - \sigma_s(k\tau)| = \lim_{s\to\infty} |\varphi(k^2\tau) - \sigma_s(k^2\tau)| = 0,$$

We need only consider the sum in equation (4.3), from $v = 1$ to $v = s$ rather than from $v = 0$ to $v = s$. If we substitute the value of $\alpha_{i,v}$ in equation (4.3) we get:

$$\left| \sum_{v=1}^{s} \sum_{i=0}^{v+1} \alpha_{i,v}[\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)] \right|$$
$$= \left| \sum_{v=1}^{s} \left[\mathop{S}_{i=0}^{v} (2-k) \frac{[\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)]}{k^{(i+1)v - \sum_i^i \alpha}} - \frac{1}{k^v} \mathop{S}_{i-1=0}^{v} \frac{\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)}{k^{iv - \sum_i^{i-1} \alpha}} \right] \right| \tag{4.4}$$

We will now take $\lim_{s\to\infty}$ of equation (4.4). Since we will in the process create a double infinite series it is convenient to rearrange the order of summation by changing the index. Rearrangement will not affect convergence because the series in question is a power series. We let $\mu = s + 1 - v$, then:

$$\lim_{s\to\infty} (2 - k) \sum_{v=1}^{s} \mathop{S}_{i=0}^{v} \frac{\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)}{k^{(i+1)v - \sum^{i} j_\alpha}}$$

$$= \lim_{s\to\infty} (2 - k) \sum_{\mu=1}^{s} \mathop{S}_{i=0}^{s+1-\mu} \frac{k^{\sum_j^i \alpha + (i+1)(s+1)} \tau^{s+1} A_{s-l+1}}{k^{(i+1)(s+1-\mu)}}$$

$$= \lim_{s\to\infty} (2 - k)\tau^{s+1} A_{s-l+1} \sum_{\mu=1}^{s} \mathop{S}_{i=0}^{s+1-\mu} k^{\sum_j^i \alpha + (i+1)\mu} = 0$$

because $0 < k < 1$ and therefore $\sum_{\mu=1}^{\infty} S_{i=0}^{\infty} k^{\sum_j^i \alpha + (i+1)\mu}$ converges. Similarly:

$$\lim_{s\to\infty} \sum_{v=1}^{s} \mathop{S}_{i-1=0}^{v} \frac{\varphi(k^{i+1}\tau) - \sigma_s(k^{i+1}\tau)}{k^{(i+1(v - \sum_j^{i-1} \alpha}} = \lim_{s\to\infty} \tau^{s+1} A_{s-1+1} \sum_{\mu=1}^{s} \mathop{S}_{i-1=0}^{s+1-\mu} k^{\sum_j^{i-1} \alpha + (i+1)\mu} = 0$$

because $0 < k < 1$ and therefore $\sum_{\mu=1}^{\infty} S_{i-1=0}^{\infty} k^{\sum_j^{i-1} \alpha + (i+1)\mu}$ converges.

Equation (4.2) therefore holds and:

$$\varphi(\tau) = \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v} \varphi(k^{i+1}\tau).$$

Theorem 3 establishes that functions of the form:

$$\varphi(t) = \sum_{i=1}^{n} P_i(t)(t - a_i)^{2\gamma_i}$$

belong to Φ', where $P_i(t)$ are polynomials, $\gamma_i \geq 0$ and $0 \leq a_i \leq 1$. Such functions are, of course, not analytic when $2\gamma_i$ are not integers. We write $2\gamma_i$ to stress the fact that we confine ourselves to real numbers. To show that $\varphi \in \Phi'$, one can see that if $0 \leq a \leq 1$, then:

$$(t - a)^{2\gamma} = [1 - (1 - (t - a)^2)]^\gamma = (1 - \tau^2)^\gamma \quad \text{where} \quad \tau^2 = 1 - (t - a)^2.$$

$(1 - \tau^2)^\gamma$ has a binomial series expansion which converges for $\gamma \geq 0$, $|\tau| \leq 1$. Since $\tau^2 = 1 - (t - a)^2$, $0 \leq a \leq 1$, it is clear that $t \in I'$ defines $\tau \in I'$, therefore by Theorem 3, $\varphi(t) = (t - a)^{2\gamma} \in \Phi'$, where $\tau^2 = 1 - (t - a)^2$, $(\frac{1}{2} < k < 1)$. Theorem 3, also establishes that $\tau^m\varphi(\tau) \in \Phi'$, m is a non-negative integer, and therefore $P(\tau)\varphi(\tau) \in \Phi'$ where $P(\tau)$ is a polynomial in τ. It is clear that if $P(t - a)$ contains only even powers* of $(t - a)$ then $P(t - a) = P^*((t - a)^2) = P^*((1 - \tau^2))$ and $P(t - a)(t - a)^{2\gamma} \in \Phi'$. If $P(t - a)$ contains only odd powers of $(t - a)$ then

$$P(t - a) = (t - a)\, P^*((t - a)^2)$$

and

$$P(t - a)(t - a)^{2\gamma} = P^*((t - a)^2)(t - a)^{2\gamma+1} \in \Phi'.$$

*Zero is assumed here to be even and not odd.

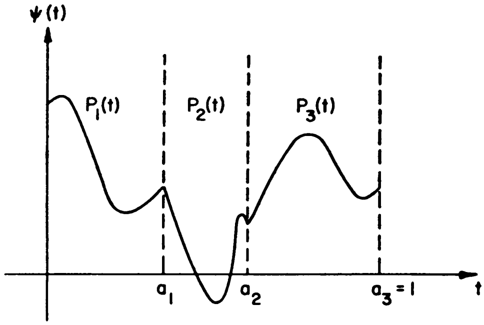

FIG. 1. Three joined polynomial segments

Since any polynomial $P(t)$ in t, can always be written as a polynomial $P(t) = \pi(t - a)$ it is clear that $P(t)(t - a)^{2\gamma} \varepsilon \Phi'$, and therefore $\varphi(t) = \sum_{i=1}^{n} P_i(t)(t - a_i)^{2\gamma_i} \varepsilon \Phi'$.

The restriction $t \varepsilon I'$ is a little too rigid. One might take $\max_{i=1,\cdots,n}(a_i - 1) \leq t \leq \min_{i=1,\cdots,n}(1 + a_i)$. In fact, since we generate values $k^{i+1}\tau_i$ we do evaluate $\varphi_i^*(\tau_i)$ in the entire region $0 < \tau_i^2 < 1 - (t - a_i)^2$ which may include negative values of t. For the purpose of general unity we prefer to restrict $t \varepsilon I'$. The series representation is of particular convenience if

$$\varphi(t) = \sum_{i=1}^{m} b_i(t - a)^{2\gamma i}$$

where the b_i are constants. In this case the representation depends on only one τ, i.e., $\tau^2 = 1 - (t - a)^2$; and we have:

$$\varphi(t) = \sum_{v=0}^{\infty} \sum_{i=0}^{v+1} \alpha_{i,v} \left[\sum_{i=1}^{m} b_i[1 - (k^{i+1}\tau)^2]^{\gamma i} \right]$$

$t \varepsilon I'$, $\frac{1}{2} < k < 1$.

An interesting class of function in Φ' are the function $\psi(t)$ defined by the polynomials $P_i(t)$, of order $n_i \geq 2$, $i = 1, \cdots, m$, in the intervals $[a_i, a_{i+1}]$, $0 = a_1 < \cdots < a_{m+1} = 1$, where $P_i(a_{i+1}) = P_{i+1}(a_{i+1})$* (see Fig. 1. where $m = 3$). To demonstrate that $\psi \varepsilon \Phi'$ we first see that $|t - a_i| = [1 - (1 - (t - a_i)^2)]^{1/2} \varepsilon \Phi'$. Then any function

$$\varphi_i(t) = P_i(t)\ |t - a_i|\ \varepsilon\ \Phi' \tag{4.5}$$

where $P_i(t)$ is a polynomial. We now show that $\psi(t)$ has a representation in the form of a linear combination of the $\varphi_i(t)$. If so $\psi(t) \varepsilon \Phi'$.

We define the polynomials $P_i'(t)$ and $P_i''(t)$ by:

$$P_i(t) = P_i(a_i) + (t - a_i)P_i'(t)$$

$$P_i'(t) = P_i'(a_{i+1}) + (t - a_{i+1})P_i''(t).$$

*Particularly interesting because many approximation procedures depend on joining polynomial segments in consecutive segments.

It is clear that $\psi(t)$ has the following representation:

$$\psi(t) = P_1(a_1) - \frac{1}{2}\sum_{i=1}^{m}[(t - a_{i+1}) - |t - a_{i+1}|]P_i'(a_{i+1})$$

$$-\frac{1}{2}\sum_{i=1}^{m}[(t - a_i)\{|t - a_{i+1}|\,P_i''(t) - P_i'(a_{i+1})\} - |t - a_i|\{(t - a_{i+1})P_i''(t) + P_i'(a_{i+1})\}].$$

This is true because:

$$A = -\frac{1}{2}\sum_{i=1}^{m}[(t - a_i)\{|t - a_{i+1}|\,P_i''(t) - P_i'(a_{i+1})\}$$

$$- |t - a_i|\,\{(t - a_{i+1})P_i''(t) + P_i'(a_{i+1})\}]$$

$$= (t - a_i)P_i'(t) + \sum_{j=1}^{i-1}(t - a_j)P_j'(a_{j+1}) \quad \text{for} \quad a_i \leq t \leq a_{i+1}$$

and

$$B = -\frac{1}{2}\sum_{i=1}^{m}[(t - a_{i+1}) + |t - a_{i+1}|]P_i'(a_{i+1})$$

$$= -\sum_{j=1}^{i-1}(t - a_{j+1})P_j'(a_{j+1}) \quad \text{for} \quad a_i \leq t \leq a_{i+1}\,.$$

therefore for $a_i \leq t \leq a_{i+1}$

$$\psi(t) = P_1(a_1) + A + B = P_1(a_1) + (t - a_i)P_i'(t) + \sum_{j=1}^{i-1}(a_{j+1} - a_j)P_j'(a_{j+1})$$

$$= P_1(a_1) + (t - a_i)P_i'(t) + \sum_{j=1}^{i-1}[P_j(a_{j+1}) - P_j(a_j)]$$

$$= (t - a_i)P_i'(t) + P_{i-1}(a_i) = (t - a_i)P_i'(t) + P_i(a_i) = P_i(t).$$