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ON THE ASYMPTOTIC SOLUTION OF NONHOMOGENEOUS ORDINARY DIFFERENTIAL EQUATIONS WITH A LARGE PARAMETER*

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1. Introduction. The determination of the asymptotic forms of the solutions to homogeneous, linear, second-order, ordinary differential equations with a large parameter has been the subject of numerous recent investigations, most of which are based on the method of R. E. Langer [1]. The emphasis on the homogeneous equation is justified, since the complementary solutions, used in the standard formula obtained by variation of parameters, readily give a particular solution of the nonhomogeneous equation. When the complementary solutions are known only to some level of approximation, careful use of the standard formula will often yield a particular solution to the same level of approximation. However, a more simple particular solution, to the same level of approximation, may exist. In [2], such a particular solution, obtained by intuitive argument, for the equation with a "transition point" is utilized. In [3] the solution of [2] is shown rigorously to be the leading term of an asymptotic series representation of the particular solution. Another approach, given in [4], to the equation considered in [2] and [3] is based on a power series expansion of the non-homogeneous term; the main disadvantage is that a simple uniformly valid solution cannot be obtained.

In this investigation, an asymptotic particular solution is obtained for the nonhomogeneous form of the equation treated in [1], which gives the result of [2] as a special case. This asymptotic particular solution has the same simplicity of form as the asymptotic complementary solutions obtained in [1]. The proof follows the line of reasoning used in [5] for the homogeneous equation with a transition point.

The equation considered is

$$\frac{d^2y}{dx^2} + p(x)\,\frac{dy}{dx} + [(k\gamma x^{\gamma-1}q(x))^2 + r(x)]y = (k\gamma x^{\gamma-1}q(x))^2 x^{\gamma(\mu-1)}g(x) \tag{1.1}$$

on the interval $0 < x < L$. The real constants are restricted to the values $\gamma \geq \epsilon$, $\mu \geq -1 + \epsilon$, where ϵ is an arbitrary, small, positive constant, independent of k, which may be taken as fixed throughout the discussion. The number k, to be considered as the large parameter, has a restricted argument $\epsilon \leq \arg k \leq \pi - \epsilon$. The coefficients are continuous and single-valued in the interval $0 < x < L$ and have the behavior**

$$p(x) = \frac{1}{x} + O(x), \qquad p'(x) = -\frac{1}{x^2} + O(1);$$

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**In this note, the notation $f(x; k) = O(\varphi(x; k))$ indicates that, for $0 \leq x \leq L - \epsilon$, $|f(x; k)| \leq C|\varphi(x; k)|$ where C is a constant independent of the parameter k.

$$q(x) = 1 + O(x^2), \quad q'(x) = O(x), \quad q''(x) = O(1); \quad r(x) = -\frac{\gamma^2\nu^2}{x^2} + O(1)$$

For definiteness, ν will have restricted argument $-\pi/2 < arg\ \nu \leq \pi/2$. The function $q(x)$ has no zeros in the interval.

Numerous other equations may be transformed into the form (1.1). Of particular interest is the equation

$$u'' + \bar{p}u' + [(k\gamma x^{\gamma-1}q)^2 + \bar{r}u] = (k\gamma x^{\gamma-1}q)^2 x^{\gamma(\mu-1)+(1-\alpha)/2} g$$

where

$$\bar{p} = \frac{\alpha}{x} + O(x), \qquad \bar{p}' = -\frac{\alpha}{x^2} + O(1), \qquad \bar{r} = \frac{\beta}{x^2} + O(1),$$

in which the constants α and/or β may be zero. This is identical to (1.1) with

$$y = x^{(\alpha-1)/2}u, \qquad p = \bar{p} + \frac{1-\alpha}{x},$$

$$r = \bar{r} + \frac{1-\alpha}{4x^2}(-1 - \alpha + 2x\bar{p}), \qquad \gamma^2\nu^2 = \left(\frac{1-\alpha}{2}\right)^2 - \beta.$$

The equation with a simple "transition point," considered in [2-5], is obtained by setting $\alpha = \beta = 0$ and $\gamma = 3/2$.

The nonhomogeneous term $g(x)$ may be complex-valued and has the behavior $g(x) = 1 + O(x^2)$, $g'(x) = O(x)$, but has no zeros in the interval. It is intuitively evident that a particular solution of (1.1) should have the behavior for $\epsilon \leq x \leq L - \epsilon$ as $|k| \to \infty$

$$y(x) \sim x^{\gamma(\mu-1)} g(x) \tag{1.2}$$

An asymptotic form of the particular solution will be sought which has this simple behavior (1.2) but which, in addition, is valid uniformly in the interval $0 \leq x \leq L - \epsilon$.

2. Complementary solutions. The asymptotic solution of the homogeneous equation, (1.1) with $g \equiv 0$, is discussed in [1]. Since the discussion in the next section of the particular solution is an extension, an argument similar to that used in [5] for obtaining the complementary solution will be briefly given.

Bessel's equation

$$\mathcal{C}_\nu''(\lambda) + \frac{1}{\lambda}\mathcal{C}_\nu'(\lambda) + \left(1 - \frac{\nu^2}{\lambda^2}\right)\mathcal{C}_\nu(\lambda) = 0 \tag{2.1}$$

with the transformation

$$\mathcal{C}_\nu(\lambda) = W(x)/\psi(x), \tag{2.2}$$

where

$$\lambda = \lambda(x) = k\gamma \int_0^x t^{\gamma-1} q(t)\, dt = kx^\gamma[1 + O(x^2)], \tag{2.3}$$

$$2\frac{\psi'(x)}{\psi(x)} = -p + \frac{\lambda'}{\lambda} - \frac{\lambda''}{\lambda'},$$

that is,

$$\psi = \left(\frac{\gamma\lambda}{\lambda' x}\right)^{-1/2} \exp\left[-\frac{1}{2}\int_0^x \left(p - \frac{1}{t}\right) dt\right], \tag{2.4}$$

becomes the equation

$$L[W] = W''(x) + pW'(x) + [(k\gamma x^{\gamma-1}q)^2 + r - Q]W(x) = 0, \tag{2.5}$$

in which

$$Q(x) = r + \nu^2 \frac{[\lambda'(x)]^2}{\lambda^2} + \frac{\psi'(x)}{\psi} + \frac{\psi''(x)}{\psi} = O(1).$$

The equation $L[U(x)] = R(x)$ has the solution

$$U(x) = W(x) + \int G(\lambda, \tau) \frac{\psi(x)\tau}{\psi(t)\tau'(t)} R(t)\, dt, \tag{2.6}$$

where

$$\lambda = \lambda(x), \qquad \tau = \lambda(t)$$

$$G(\lambda, \tau) = \frac{\pi i}{2} [J_\nu(\lambda)H_\nu^{(1)}(\tau) - J_\nu(\tau)H_\nu^{(1)}(\lambda)].$$

Therefore (1.1) with $g \equiv 0$, which may be written in the form $L[y(x)] = -Q(x)y(x)$, is equivalent to the Volterra integral equation

$$y(x) = W(x) - \int_0^x G(\lambda, \tau) \frac{\psi(x)\tau}{\psi(t)\tau'(t)} Q(t)y(t)\, dt. \tag{2.7}$$

Consider the case $W(x) = \psi(x)J_\nu(\lambda)$. If the solution of (1.1) is written as

$$y_1(x) = \psi(x)J_\nu(\lambda)f(x), \tag{2.8}$$

then (2.7) immediately yields

$$f(x) = 1 - \int_0^x \frac{G(\lambda, \tau)J_\nu(\tau)\tau^{-1+2/\gamma}}{J_\nu(\lambda)} \left[\frac{Q(t)f(t)\tau^{2(\gamma-1)/\gamma}}{(\tau')^2}\right]\tau'\, dt. \tag{2.6}$$

Hence

$$|f(x)| \le 1 + |f(x)|_{\max}\delta_1\,, \qquad |f(x) - 1| \le |f(x)|_{\max}\delta_1\,,$$

where "max" denotes the maximum in the interval $0 \le x \le L - \epsilon$, and

$$\delta_1 = \left|\frac{Q(x)\lambda^{2(\gamma-1)/\gamma}}{(\lambda')^2}\right|_{\max} \left\{\int_0^\lambda \left|\frac{G(\lambda, \tau)J_\nu(\tau)\tau^{-1+2/\gamma}}{J_\nu(\lambda)}\, d\tau\right|\right\}_{\max}.$$

In particular

$$|f(x)|_{\max} \le 1 + |f(x)|_{\max}\delta_1$$

so that (if $\delta_1 < 1$)

$$|f(x)|_{\max} \le (1 - \delta_1)^{-1}.$$

Thus,

$$|f(x) - 1| \le \delta_1/(1 - \delta_1) \tag{2.9}$$

The number δ_1 can be computed for a specific problem; of more interest is its behavior with respect to k. It is easily seen that

$$\frac{\lambda^{2(\gamma-1)/\gamma}}{(\lambda')^2} = O(k^{-2/\gamma}).$$

Furthermore, for $\operatorname{Re}\nu \geq \epsilon$, $0 \leq |\tau| \leq |\lambda|$, $\arg\tau = \arg\lambda$, and $\epsilon \leq \arg\lambda \leq \pi - \epsilon$, the following inequality holds

$$\left|\frac{G(\lambda, \tau)J_\nu(\tau)}{J_\nu(\lambda)}\right| \leq \frac{C}{1 + |\tau|}.$$

For $\gamma > 2$,

$$\int_0^\lambda \left|\frac{\tau^{-1+2/\gamma}}{1 + |\tau|}\, d\tau\right| \leq \int_0^1 |\tau|^{-1+2/\gamma} d\,|\tau| + \int_1^\infty |\tau|^{-2+2/\gamma} d\,|\tau| = \frac{\gamma^2}{2(\gamma - 2)}.$$

Thus,

$$\int_0^\lambda \left|\frac{\tau^{-1+2/\gamma}}{1 + |\tau|}\, d\tau\right| \leq \begin{cases} \dfrac{\gamma^2}{2(\gamma - 2)} & \text{for} \quad \gamma > 2, \\ \log(1 + |\lambda|) & \text{for} \quad \gamma = 2, \\ \dfrac{|\lambda|^{-1+2/\gamma}}{-1 + 2/\gamma} & \text{for} \quad \epsilon \leq \gamma < 2, \end{cases}$$

from which follows the result

$$\delta_1 = O(k_\gamma^{-1}), \tag{2.10}$$

where

$$k_\gamma = \begin{cases} |k|^{2/\gamma} & \text{for} \quad \gamma > 2, \\ |k|/\log|k| & \text{for} \quad \gamma = 2, \\ |k| & \text{for} \quad \epsilon \leq \gamma < 2. \end{cases} \tag{2.11}$$

The estimate also holds for $\operatorname{Re}\nu = 0$. With Eqs. (2.8), (2.9), and (2.10), it follows that one solution of (1.1) with $g \equiv 0$ is of the form

$$y_1(x) = \psi(x)J_\nu(\lambda)[1 + O(k_\gamma^{-1})]. \tag{2.12}$$

Similarly the second solution of (1.1) is found to be

$$y_2(x) = \psi(x)H_\nu^{(1)}(\lambda)[1 + O(k_\gamma^{-1})]. \tag{2.13}$$

For the derivative, it may be shown that

$$y_1'(x) = \frac{d}{dx}\,[\psi(x)J_\nu(\lambda)][1 + O(k_\gamma^{-1})].$$

However,

$$\left|\frac{\psi'(x)J_\nu(\lambda)}{\psi\lambda'(x)J_\nu'(\lambda)}\right| \leq \begin{cases} C\,|k|^{-2/\gamma} & \text{for} \quad \gamma \geq 2, \\ C\,|k|^{-1} & \text{for} \quad \gamma \leq 2. \end{cases}$$

Hence the derivative of ψ makes a contribution that is of higher order than that of the derivative of $J_\nu(\lambda)$, and so

$$y_1'(x) = \psi(x)\lambda'(x)J_\nu'(\lambda)[1 + O(k_\gamma^{-1})], \tag{2.14}$$

with the similar second solution

$$y_2'(x) = \psi(x)\lambda'(x)\,\frac{d}{d\lambda}\,H_\nu^{(1)}(\lambda)[1 + O(k_\gamma^{-1})]. \tag{2.15}$$

3. Particular solution. For $g(x) \not\equiv 0$, the integral Eq. (2.7) becomes

$$y(x) = \int^{x} G(\lambda, \tau) \frac{\psi(x)}{\psi(t)} t^{\gamma(\mu-1)} g(t) \tau\tau'(t)\, dt - \int^{x} G(\lambda, \tau) \frac{\psi(x)\tau}{\psi(t)\tau'(t)} Q(t) y(t)\, dt \tag{3.1}$$

The lower limit for the integrals containing $J_\nu(\tau)$ will be chosen as $x = 0$, while the lower limit for the integrals containing $H_\nu^{(1)}(\tau)$ will be chosen as $x = L - \epsilon_1$ where ϵ_1 is a fixed constant in the range $0 < \epsilon_1 < \epsilon$.

Integration by parts of the nonhomogeneous term of (3.1), denoted by I, gives

$$I = \frac{\pi i}{2} \psi(x) J_\nu(\lambda) \left\{ \left(\frac{x^\gamma}{\lambda}\right)^{\mu-1} \frac{g(x)}{\psi(x)} \int_{\lambda(L-\epsilon_1)}^{\lambda(x)} H_\nu^{(1)}(\tau)\, d\tau \right.$$
$$\left. - \int_{\lambda(L-\epsilon_1)}^{\lambda(x)} \left(\int_{\lambda(L-\epsilon_1)}^{\tau} H_\nu^{(1)}(\tau)\tau^\mu\, d\tau \right) \frac{d}{dt} \left[\left(\frac{t^\gamma}{\tau}\right)^{\mu-1} \frac{g(t)}{\psi(t)} \right] dt \right\} - \frac{\pi i}{2} \psi(x) H_\nu^{(1)} \{ \cdots \}. \tag{3.2}$$

But, since all functions are uniformly continuous on the interval $L - \epsilon \le x \le L - \epsilon_1$,

$$\frac{d}{dt} \left[\left(\frac{t^\gamma}{\tau}\right)^{\mu-1} \frac{g(t)}{\psi(t)} \right] \frac{\tau^{1-2/\gamma}}{\tau'(t)} = O(|k|^{1-\mu-2/\gamma})$$

and for $\operatorname{Re} \nu \ge 0$, $\mu \ge -1 + \epsilon$.

$$\frac{\left| \int_{+i\infty}^{\lambda} \left| \int_{+i\infty}^{\tau} H_\nu^{(1)}(\tau)\tau^\mu\, d\tau \right| \cdot \tau^{-1+2/\gamma}\, d\tau \right|}{\left| \int_{+i\infty}^{\lambda} H_\nu^{(1)}(\tau)\tau^\mu\, d\tau \right|} \le \begin{cases} C(|\lambda|_{\max})^{-1+2/\gamma} & \text{for } \epsilon \le \gamma \le 2, \\ C & \text{for } \gamma \le 2, \end{cases}$$

with a similar estimate for the integral containing $J_\nu(\tau)$. Hence Eq. (3.2) is of the form

$$I = V(x)[1 + O(e^{-|k|(L-\epsilon_1-x)}) + O(k_{\gamma 3}^{-1})], \tag{3.3}$$

where

$$k_{\gamma 3} = \begin{cases} |k|^{2/\gamma} & \text{for } \gamma \ge 2, \\ |k| & \text{for } \epsilon \le \gamma \le 2, \end{cases} \tag{3.4}$$

$$V(x) = \left(\frac{x^\gamma}{\lambda}\right)^{\mu-1} g(x) F_{\mu,\nu}(\lambda), \tag{3.5}$$

with

$$F_{\mu,\nu}(\lambda) \equiv \frac{\pi i}{2} J_\nu(\lambda) \int_{+i\infty}^{\lambda} H_\nu^{(1)}(\tau)\tau^\mu\, d\tau - H_\nu^{(1)}(\lambda) \int_0^\lambda J_\nu(\tau)\tau^\mu\, d\tau$$
$$= s_{\mu,\nu}(\lambda) - 2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right) e^{(\mu-\nu+1)\pi i/2} J_\nu(\lambda)$$
$$= S_{\mu,\nu}(\lambda) - 2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right) \Gamma\left(\frac{\mu+\nu+1}{2}\right) e^{(1-\nu)\pi i/2} \cos\left(\frac{\mu-\nu}{2}\pi\right) H_\nu^{(1)}(\lambda),$$

$s_{\mu,\nu}$ and $S_{\mu,\nu}$ being Lommel's functions discussed in [6].

The function $F_{\mu,\nu}(\lambda)$ is bounded at $\lambda = 0$ for $\mu \ge -1 + \epsilon$ and $\nu \ge 0$, and has the behavior

$$F_{\mu,\nu}(\lambda) \sim \lambda^{\mu-1}$$

as $|\lambda| \to \infty$ while $\epsilon \leq \arg \lambda \leq \pi - \epsilon$. Hence the nonhomogeneous term of (3.1) given by (3.3) has the desired behavior (1.2) of a particular solution. Furthermore the constants of integration in (3.1), leading to those of (3.6), were chosen so that (3.3) would be bounded at $x = 0$.

Therefore, the particular solution of (1.1), denoted by y_3, will be sought in the form

$$y_3(x) = V(x)h(x). \tag{3.7}$$

Two numbers are defined:

$$\delta_2 = \left|\frac{I}{V} - 1\right|_{\max},$$

$$\delta_3 = \left(\int^x \left|G(\lambda, \tau)\, \frac{\psi(x)\tau}{\psi(t)\tau'(t)}\, Q(t)\, V(t)\right| dt/|V(x)|\right)_{\max}.$$

The form (3.7) when substituted into (3.1) yields the result

$$|h| \leq 1 + \delta_2 + |h|_{\max}\delta_3$$

from which follows (for $\delta_3 < 1$) that

$$|h|_{\max} \leq \frac{1 + \delta_2}{1 - \delta_3}.$$

We also have the result

$$|h - 1| \leq \delta_2 + |h|_{\max}\delta_3 ,$$

which, with the result for $|h|_{\max}$, gives

$$|h - 1| \leq \frac{\delta_2 + \delta_3}{1 - \delta_3}. \tag{3.8}$$

The behavior of δ_2 follows immediately from (3.3). If ϵ_1 is chosen to be

$$\epsilon_1 = \epsilon - |k|^{-1} \log |k|,$$

which will be positive for sufficiently large $|k|$, one obtains

$$\delta_2 = O(k^{-1}_{\gamma 3}). \tag{3.9}$$

For δ_3 one obtains

$$\delta_3 \leq \frac{\left|\psi\left(\frac{x^\gamma}{\lambda}\right)^{\mu-1} g\right|_{\max}}{\left|\psi\left(\frac{x^\gamma}{\lambda}\right)^{\mu-1} g\right|_{\min}} \cdot |Q|_{\max} \cdot \left|\frac{\lambda^{2-2/\gamma}}{(\lambda')^2}\right|_{\max} \times \int^\lambda \left|G(\lambda, \tau)\tau^{-1+2/\gamma}\, \frac{F_{\mu,\nu}(\tau)}{F_{\mu,\nu}(\lambda)}\, d\tau\right|.$$

But

$$\frac{\lambda^{2-2/\gamma}}{(\lambda')^2} = O(k^{-2/\gamma})$$

and

$$\int^\lambda \left|G(\lambda, \tau)\tau^{-1+2/\gamma}\, \frac{F_{\mu,\nu}(\tau)}{F_{\mu,\nu}(\lambda)}\, d\tau\right| \leq \begin{cases} C|\lambda|^{-2+2/\gamma} & \text{for} \quad \epsilon \leq \gamma \leq 1, \\ C & \text{for} \quad 1 \leq \gamma. \end{cases}$$

Hence

$$\delta_3 = \begin{cases} O(|k|^{-2}) & \text{for} \quad \epsilon \le \gamma \le 1, \\ O(|k|^{-2/\gamma}) & \text{for} \quad 1 \le \gamma, \end{cases}$$

or, conservatively for $\gamma < 2$,

$$\delta_3 = O(k_{\gamma 3}^{-1}). \tag{3.10}$$

From the results (3.7)-(3.16), it is seen that the particular solution of (1.1), which is bounded at $x = 0$ and has the behavior (1.2), is indeed

$$y_3(x) = x^{\gamma(\mu-1)} g(x) \lambda^{1-\mu} F_{\mu,\nu}(\lambda)[1 + O(k_{\gamma 3}^{-1})]. \tag{3.11}$$

By similar reasoning the derivative may be shown to be

$$y_3'(x) = V'(x)[1 + O(k_{\gamma 3}^{-1})].$$

However, for $\mu \neq 1$, a simplification may be obtained

$$V'(x) = \frac{d}{dx}[x^{\gamma(\mu-1)} g(x)] \frac{\lambda^{2-\mu}}{1-\mu} F_{\mu,\nu}'(\lambda)[1 + u_1(x)u_2(\lambda)],$$

where

$$u_1(x) = (\mu - 1)\frac{\lambda'}{\lambda} \frac{x^{\gamma(\mu-1)} g(x)}{\dfrac{d}{dx}[x^{\gamma(\mu-1)} g(x)]} - 1, \qquad u_2(\lambda) = \frac{\lambda^{\mu-1} \dfrac{d}{d\lambda}[\lambda^{1-\mu} F_{\mu,\nu}(\lambda)]}{F_{\mu,\nu}'(\lambda)}$$

But, since

$$|u_1(x)| = |O(x^2)| \le C\,|k|^{-2/\gamma}\,|\lambda|^{2/\gamma},$$

and

$$|u_2(\lambda)| \le \frac{C}{1 + |\lambda|^2}$$

for $|\lambda| \ge 0$ and $\epsilon \le \arg \lambda \le \pi - \epsilon$, it follows that

$$u_1(x)u_2(\lambda) = \begin{cases} O(|k|^{-2/\gamma}) & \text{for} \quad \gamma \ge 1, \\ O(|k|^{-2}) & \text{for} \quad \epsilon \le \gamma \le 1. \end{cases}$$

Hence, the derivative of the particular solution has the form, for $\mu \neq 1$,

$$y_3'(x) = \frac{d}{dx}[x^{\gamma(\mu-1)} g(x)] \frac{\lambda^{2-\mu}}{1-\mu} F_{\mu,\nu}'(\lambda)[1 + O(k_{\gamma 3}^{-1})] \tag{3.12}$$

Since

$$\lambda^{2-\mu} F_{\mu,\nu}'(\lambda) \sim 1 - \mu$$

as $|\lambda| \to \infty$ while $\epsilon \le \arg \lambda \le \pi - \epsilon$, the derivative approaches the value

$$y_3'(x) \sim \frac{d}{dx}[x^{\gamma(\mu-1)} g(x)]$$

in the interval $\epsilon \le x \le L - \epsilon$ as $|k| \to \infty$.

4. Application. Similar to the complementary solutions (2.12)–(2.15), the particular solution (3.11) and its derivative (3.12) are in the form of a product of an easily computed function of x and a function of the transformation variable λ. Furthermore, certain results of boundary value problems of applied interest, as in [7], may also be obtained in the product form. For example, consider the general (asymptotic) solution of (1.1)

$$y(x) = A\psi J_\nu(\lambda) + B\psi H_\nu^{(1)}(\lambda) + [x^{\gamma(\mu-1)}g(x)]\lambda^{1-\mu}F_{\mu,\nu}(\lambda). \tag{4.1}$$

If it is prescribed that $y(L - \epsilon)$ be bounded (so $A = 0$) and that $y(x_0) = 0$, then $y'(x_0)$ is given by

$$y'(x_0) = \frac{d\lambda(x_0)}{dx}[x_0^{\gamma(\mu-1)}g(x_0)]\left[-\frac{\frac{d}{d\lambda}H_\nu^{(1)}(\lambda_0)}{H_\nu^{(1)}(\lambda_0)}\lambda_0^{1-\mu}F_{\mu,\nu}(\lambda_0) + \lambda_0^{1-\mu}F'_{\mu,\nu}(\lambda_0)\right] \cdot [1 + v_1(x_0)v_2(\lambda_0)], \tag{4.2}$$

where $\lambda_0 = \lambda(x_0)$, and where

$$v_1(x) = \frac{\lambda\frac{d}{dx}[x^{\gamma(\mu-1)}g(x)]}{\frac{d\lambda}{dx}[x^{\gamma(\mu-1)}g(x)](1-\mu)} - 1, \qquad v_2(\lambda) = \frac{1}{1 - \frac{\frac{d}{d\lambda}H_\nu^{(1)}(\lambda)F_{\mu,\nu}(\lambda)}{H_\nu^{(1)}(\lambda)F'_{\mu,\nu}(\lambda)}}.$$

But

$$v_1(x) = O(x^2), \qquad |v_2(\lambda)| \le \frac{C}{1+|\lambda|}.$$

and hence

$$v_1(x)v_2(\lambda) = O(k_{\gamma3}^{-1}). \tag{4.3}$$

The Bessel and Lommel's functions required for the solution (4.1) are generally untabulated, although their behavior has been thoroughly examined [6]. However, for the case arg $\lambda = 3\pi/4$, which occurs in various applied problems, the real and imaginary parts of $J_\nu(\lambda)$, $H_\nu^{(1)}(\lambda)$, $\lambda^{1-\mu}F_{\mu,\nu}(\lambda)$, and $\lambda^{2-\mu}F'_{\mu,\nu}(\lambda)$ are tabulated in [7] for $\mu = 0, 1, 2$ and $\nu = 0, 0.1, \cdots, 1.0$ for $|\lambda| = 1, 2, \cdots, 10$.

5. Conclusion. The solution (4.1) and such results as (4.2) have a simplicity of form which yields considerable insight into the behavior of the exact solution of (1.1). Although from a computational standpoint, it may seem that the use of the solution (4.1) offers little advantage over a direct numerical integration of (1.1) since the evaluation of the Bessel and Lommel's functions needed in (4.1) require, in general, an almost equal numerical effort. However, an evaluation of the Bessel and Lommel's functions for a set of values of the parameters λ, ν, and μ provides the approximate solution for a class of equations (1.1); in contrast the direct numerical integration must be repeated for every change in the coefficients p, q, r, and g. Furthermore, difficulties are encountered with direct numerical integration in the neighborhood of $x = 0$ and when $|k|$ becomes very large. On the other hand, the asymptotic solution becomes invalid for small values of $|k|$; the determination of the minimum allowable $|k|$ for a prescribed accuracy requires the tedious numerical chore of evaluating δ_1, δ_2, and δ_3. Therefore the asymptotic and the numerical integration methods, with their respective advantages and disadvantages, offer complementary, not directly competing, means of evaluating solutions of differential equations.

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