

WAVE PROPAGATION IN THE LINEAR THEORY OF VISCOELASTICITY*

BY

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1. Introduction. Recently Herrera and Gurtin [1] have shown that in the linear theory of viscoelasticity the speed of propagation U of an *acceleration wave* is a solution of the eigenvalue problem

$$(G_{ijkl}(0)n_j n_l - \rho U^2 \delta_{ik})\beta_k = 0. \quad (1.1)$$

Here $G_{ijkl}(0)$ is the initial value of the relaxation function $G_{ijkl}(t)$; ρ is the mass density; \mathbf{n} is a unit vector in the direction of propagation of the wave.

In this paper we show that the eigenvalue problem (1.1) governs not only the speed of propagation of acceleration waves but also the propagation speeds of *shocks and all higher order waves*.

Finally we remark that the theory presented here is not empty. Indeed, within the context of linear viscoelasticity theory, Chu [2] has constructed a solution which has a first order discontinuity (shock). In addition, solutions which contain discontinuities of arbitrary order N are easily exhibited.

Notation. Henceforth, \mathcal{R} denotes an open region of Euclidean three dimensional space; $\mathbf{x} = (x_1, x_2, x_3)$ is a generic point of \mathcal{R} ; \mathcal{O} is a regular subregion of \mathcal{R} (called a *part* of \mathcal{R}); $\partial\mathcal{O}$ is the boundary of \mathcal{O} ; \mathbf{n} is the outward unit normal to $\partial\mathcal{O}$. The scalar t denotes the time. Given a function g of position and time we write $g^{(m)}$ for its m th time derivative, i.e., $g^{(m)}(\mathbf{x}, t) = \partial^m g(\mathbf{x}, t) / \partial t^m$.

Letters in boldface denote vectors. We shall, for the most part, use indicial notation and Cartesian tensors. Thus subscripts have the range (1, 2, 3); summation over repeated indices is understood; subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate; δ_{ij} designates Kronecker's delta. We shall sometimes find it convenient to use the vector cross product $\mathbf{x} \times \mathbf{y}$ and inner product $\mathbf{x} \cdot \mathbf{y}$.

2. The equations of linear viscoelasticity. The fundamental system of field equations for the linear theory of viscoelasticity consists of:

(i) the *constitutive relation*

$$\sigma_{ij}(\mathbf{x}, t) = \int_{-\infty}^t G_{ijkl}(\mathbf{x}, t-s) d\gamma_{kl}(\mathbf{x}, s), \quad (2.1)$$

where

$$G_{ijkl} = G_{ijlk} = G_{jikl};$$

(ii) the *strain displacement relation*

$$2\gamma_{ij}(\mathbf{x}, t) = u_{i,j}(\mathbf{x}, t) + u_{j,i}(\mathbf{x}, t); \quad (2.2)$$

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(iii) the law of balance of linear momentum

$$\frac{d}{dt} \int_{\mathcal{P}} \rho(\mathbf{x}) u_i^{(1)}(\mathbf{x}, t) d\mathbf{x} = \int_{\mathcal{P}} \rho(\mathbf{x}) f_i(\mathbf{x}, t) d\mathbf{x} + \int_{\partial \mathcal{P}} \sigma_{ji}(\mathbf{x}, t) \eta_j(\mathbf{x}) d\mathbf{x}. \quad (2.3)$$

Here ρ is the mass density, while u_i , γ_{ij} , σ_{ji} , G_{ijkl} , and f_i are respectively the Cartesian components of the displacement vector \mathbf{u} , the infinitesimal strain tensor $\boldsymbol{\gamma}$, the symmetric* stress tensor $\boldsymbol{\sigma}$, the relaxation function \mathbf{G} , and the body force density vector \mathbf{f} . The first two equations are assumed to hold at every $(\mathbf{x}, t) \in \mathcal{R} \times (-\infty, \infty)$; the third holds for every time t and every part \mathcal{P} of \mathcal{R} . We write the law of balance of linear momentum in integral form so as to include the possibility of discontinuous velocity and stress fields.

Finally we adjoin the *initial condition*

$$u(\mathbf{x}, t) = 0 \text{ whenever } (\mathbf{x}, t) \in \mathcal{R} \times (-\infty, 0). \quad (2.4)$$

3. Viscoelastic waves of order N . Henceforth S_t ($0 \leq t < \infty$) denotes a one-parameter family of surfaces with $S_t \subset \mathcal{R}$; Σ_s is the hypersurface

$$\Sigma_s = \{(\mathbf{x}, t) | \mathbf{x} \in S_t, 0 \leq t < \infty\}; \quad (3.1)$$

$\omega_{\mathbf{x}}$, for each \mathbf{x} , is the point set

$$\omega_{\mathbf{x}} = \{t | \mathbf{x} \in S_t\}; \quad (3.2)$$

$N \geq 1$ is a fixed integer.

By a *viscoelastic wave of order N* we mean a solution \mathbf{u} , $\boldsymbol{\gamma}$, $\boldsymbol{\sigma}$ of (2.1), (2.2), (2.3), and (2.4) such that:

- (A.1) The displacement \mathbf{u} is $N - 1$ times continuously differentiable on $\mathcal{R} \times (-\infty, \infty)$; however its N th order derivatives have jump discontinuities across Σ_s but are continuous everywhere else on $\mathcal{R} \times (-\infty, \infty)$. Moreover, each of these N th order derivatives is bounded on every compact subset of $\mathcal{R} \times (-\infty, \infty)$.
- (A.2) The relaxation function \mathbf{G} is N times continuously differentiable on $\mathcal{R} \times [0, \infty)$, while the body force \mathbf{f} and the density ρ are continuous and, if $N \geq 3$, $N - 2$ times continuously differentiable on their respective domains $\mathcal{R} \times (-\infty, \infty)$ and \mathcal{R} .
- (A.3) The hypersurface Σ_s is smooth and orientable.
- (A.4) For each \mathbf{x} , the set $\omega_{\mathbf{x}}$ has measure zero.

Condition (A.1) is the requirement that Σ_s be a singular surface of order N in the sense of Duhem and Hadamard.** Given a fixed point $(\mathbf{x}, t) \in \Sigma_s$, condition (A.3) implies the existence of a normal \mathbf{v} to Σ_s at (\mathbf{x}, t) . Such a vector is

$$\mathbf{v} \equiv (\mathbf{n}, -U) \equiv (n_1, n_2, n_3, -U), \quad U \geq 0,$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit normal to S_t at the point \mathbf{x} . The vector \mathbf{n} is called the *direction of propagation*; the number U is called the *speed of propagation*. For $U \neq 0$

*The symmetry of the stress tensor insures that the angular momentum is balanced.

**See Truesdell and Toupin [3], §187.

we call any non-zero vector, which is parallel to the jump* $[u^{(N)}(\mathbf{x}, t)]$, a *wave axis*. Roughly speaking, condition (A.4) asserts that the surface S_t , as it progresses, does not pass through a given point \mathbf{x} too many times.

Finally we use the following terminology: If $N = 1$ we call the wave a *shock wave***; if $N = 2$ we call the wave an *acceleration wave*.

4. Shock waves. Our first theorem shows that the speed of propagation of a shock wave is governed by the same eigenvalue problem as that which arises in the study of acceleration waves.

THEOREM 1. *At a fixed point (\mathbf{x}, t) the speed of propagation U of a viscoelastic shock wave, whose direction of propagation is \mathbf{n} , satisfies the eigenvalue problem*

$$(G_{ijkl}(\mathbf{x}, 0)n_j n_l - \rho(\mathbf{x})U^2 \delta_{ik})\beta_k = 0. \quad (4.1)$$

Moreover, the eigenvector β is an axis of the wave.

Proof. The integral statement (2.3) of the law of balance of momentum yields the well known jump condition

$$-\rho U[u_i^{(1)}] = [\sigma_{ij}]n_j. \quad (4.2)$$

If we integrate the constitutive relation (2.1) by parts and use the strain-displacement relation (2.2) together with the initial condition (2.4), we find that

$$\sigma_{ij}(\mathbf{x}, t) = G_{ijkl}(\mathbf{x}, 0)u_{k,l}(\mathbf{x}, t) + \int_0^t G_{ijkl}^{(1)}(\mathbf{x}, t-s)u_{k,l}(\mathbf{x}, s) ds. \quad (4.3)$$

Upon substituting (4.3) into (4.2) we arrive at

$$-\rho(\mathbf{x})U[u_i^{(1)}(\mathbf{x}, t)] = \left[G_{ijkl}(\mathbf{x}, 0)u_{k,l}(\mathbf{x}, t) + \int_0^t G_{ijkl}^{(1)}(\mathbf{x}, t-s)u_{k,l}(\mathbf{x}, s) ds \right] n_j. \quad (4.4)$$

From Maxwell's Theorem† we know that the jump in the four dimensional gradient‡ of the displacement \mathbf{u} is parallel to the hypersurface normal \mathbf{v} . Hence there exists a vector β such that

$$\left(\left[\frac{\partial u_i}{\partial x_1} \right], \left[\frac{\partial u_i}{\partial x_2} \right], \left[\frac{\partial u_i}{\partial x_3} \right], \left[\frac{\partial u_i}{\partial t} \right] \right) = \beta_i (n_1, n_2, n_3, -U), \quad (4.5)$$

or equivalently,

$$\left[\frac{\partial u_i}{\partial x_j} \right] = \beta_i n_j, \quad (4.6)$$

$$\left[\frac{\partial u_i}{\partial t} \right] = -\beta_i U. \quad (4.7)$$

*We use the usual notation for the jump $[g]$ of a function g across Σ_S . That is

$$[g(\mathbf{x}, t)] = \lim_{(\mathbf{x}^+, t^+) \rightarrow (\mathbf{x}, t)} g(\mathbf{x}^+, t^+) - \lim_{(\mathbf{x}^-, t^-) \rightarrow (\mathbf{x}, t)} g(\mathbf{x}^-, t^-),$$

where (\mathbf{x}^+, t^+) and (\mathbf{x}^-, t^-) lie on opposite sides of Σ_S .

**Actually, for $N = 1$, it is customary to call the wave a shock wave if $[u^{(1)} \cdot \mathbf{n}] \neq 0$, a vortex sheet if $[u^{(1)} \cdot \mathbf{n}] = 0$. We do not make this distinction.

†See [3], §175.

‡By the four dimensional gradient we mean the operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3, \partial/\partial t)$.

Thus (4.6) and (4.7), when substituted into (4.4), yield

$$\rho(\mathbf{x})U^2\beta_i = G_{i,jkl}(\mathbf{x}, 0)n_jn_l\beta_k + \left[\int_0^t G_{i,jkl}^{(1)}(\mathbf{x}, t-s)u_{k,l}(\mathbf{x}, s) ds \right] n_j. \quad (4.8)$$

Thus the proof reduces to showing that the jump in the integral in (4.8) vanishes. That such is the case is an immediate consequence of (2.1), (A.1), (A.2), (A.4), and the following lemma.

LEMMA. *Let g and k be functions on $\mathcal{R} \times (-\infty, \infty)$ with the following properties:*

- (a) g is continuous;
- (b) k is continuous everywhere except for a possible jump discontinuity across Σ_s ;
- (c) k is bounded on every compact subset of $\mathcal{R} \times (-\infty, \infty)$.

Moreover assume (A.4) holds. Then the function A defined by

$$A(\mathbf{x}, t) = \int_0^t g(\mathbf{x}, t-s)k(\mathbf{x}, s) ds$$

is continuous on $\mathcal{R} \times (-\infty, \infty)$.

Proof. If we define

$$B(\mathbf{x}, t, s) = g(\mathbf{x}, t-s)k(\mathbf{x}, s), \quad (4.9)$$

then

$$A(\mathbf{x}^*, t^*) - A(\mathbf{x}, t) = \int_0^{t^*} B(\mathbf{x}^*, t^*, s) ds + \int_t^{t^*} B(\mathbf{x}^*, t^*, s) ds - \int_0^t B(\mathbf{x}, t, s) ds. \quad (4.10)$$

We now choose $(\mathbf{x}, t) \in \mathcal{R} \times (-\infty, \infty)$ arbitrarily, pick $T > 0$ such that $(-T < t < T)$, and let Ω denote a closed sphere centered at \mathbf{x} and contained in \mathcal{R} . Further, we choose $(\mathbf{x}^*, t^*) \in \Omega \times (-T, T)$. By hypothesis, $B(\mathbf{x}^*, t^*, s)$ is bounded for

$$(\mathbf{x}^*, t^*, s) \in \Omega \times [-T, T] \times [0, t].$$

Hence the second integral in (4.10) tends to zero as $(\mathbf{x}^*, t^*) \rightarrow (\mathbf{x}, t)$. Next, by (4.9) and our assumptions regarding g , k , and $\omega_{\mathbf{x}}$, we find that

$$B(\mathbf{x}^*, t^*, s) \rightarrow B(\mathbf{x}, t, s) \quad \text{as } (\mathbf{x}^*, t^*) \rightarrow (\mathbf{x}, t)$$

for almost every $s \in [0, t]$. This fact and the boundedness of B , when used in conjunction with Lebesgue's Dominated Convergence Theorem, imply that

$$\int_0^t B(\mathbf{x}^*, t^*, s) ds \rightarrow \int_0^t B(\mathbf{x}, t, s) ds$$

and hence $A(\mathbf{x}^*, t^*) \rightarrow A(\mathbf{x}, t)$. This completes the proof.

5. Acceleration and higher order waves. Our main result is contained in the next theorem.

THEOREM 2. *The speed of propagation U of a viscoelastic wave is independent of the order N of the wave and hence satisfies the eigenvalue problem (4.1), where again β is an axis of the wave.*

Proof. By Theorem 1 it suffices to assume $N \geq 2$. For such a choice of N , (A.1) implies the $N - 1$ compatibility relations

$$\left[\frac{\partial}{\partial x_s} (u_{i,jk \dots l}^{(m)}) \right] = \left[\frac{\partial}{\partial t} (u_{i,jk \dots ls}^{(m-1)}) \right], \quad (5.2)$$

where $jk \dots l$ represent q indices, $jk \dots ls$ represent $q + 1$ indices, and $q + m = N - 1$.

Using Maxwell's Theorem for the jump in the four-dimensional gradient, we find that for each integer m ($1 \leq m \leq N - 1$) there exists an m th order tensor ${}^{(m)}\mathfrak{g}$ whose components satisfy

$$\left[\frac{\partial}{\partial x_s} u_{i,jk \dots l}^{(m)} \right] = {}^{(N-m)}\beta_{ijk \dots ls} n_s \quad (5.3)$$

and

$$\left[\frac{\partial}{\partial t} u_{i,jk \dots l}^{(m)} \right] = -{}^{(N-m)}\beta_{ijk \dots l} U. \quad (5.4)$$

Thus (5.2), (5.3), and (5.4) imply

$$-{}^{(m+1)}\beta_{ijk \dots ls} U = {}^{(m)}\beta_{ijk \dots ls} n_s, \quad (5.5)$$

and hence

$${}^{(m+1)}\beta_{ijk \dots ls} (-U)^m = \beta_{i,n_j n_k \dots n_l n_s} (\beta_i \equiv {}^{(1)}\beta_i), \quad (5.6)$$

which, because of (5.3), yields

$$(-U)^{N-m-1} [u_{i,jk \dots ls}^{(m)}] = \beta_{i,n_j n_k \dots n_l n_s}, \quad (5.7)$$

where $jk \dots ls$ indicate $N - m$ indices.

Next, since $N \geq 2$, (2.1), (2.2), and (2.3) imply the displacement equations of motion

$$\{G_{ijkl}(\mathbf{x}, 0)u_{k,l}(\mathbf{x}, t)\}_{,i} + \left\{ \int_0^t G_{ijkl}^{(1)}(\mathbf{x}, t-s)u_{k,l}(\mathbf{x}, s) ds \right\}_{,i} \\ + \rho(\mathbf{x})f_i(\mathbf{x}, t) = \rho(\mathbf{x})u_i^{(2)}(\mathbf{x}, t). \quad (5.8)$$

We now differentiate (5.8) $N - 2$ times with respect to position, take the jump in the resulting equation, and use (A.1), (A.2), and the lemma to verify that

$$G_{ijkl}(\mathbf{x}, 0)[u_{k,lipq \dots r}(\mathbf{x}, t)] = \rho(\mathbf{x})[u_{i,pq \dots r}^{(2)}(\mathbf{x}, t)]. \quad (5.9)$$

Consequently, (5.7) and (5.9) imply

$$G_{ijkl}(\mathbf{x}, 0)\beta_k n_i n_j n_p n_q \dots n_r = U^2 \rho(\mathbf{x})\beta_i n_p n_q \dots n_r. \quad (5.10)$$

Thus, if we multiply (5.10) by $n_p n_q \dots n_r$ and use the fact that $n_i n_i = 1$, we recover the eigenvalue problem (4.1) which completes the proof.

6. Longitudinal and transverse waves. We call a viscoelastic wave *longitudinal* if its direction is parallel to its axis, i.e., if

$$\mathbf{n} \times \mathfrak{g} = 0, \quad (6.1)$$

transverse if its direction is perpendicular to its axis, i.e., if

$$\mathbf{n} \cdot \mathfrak{g} = 0. \quad (6.2)$$

THEOREM 3. *Given a viscoelastic wave of order N , then the following three statements are equivalent:*

- (i) *the wave is longitudinal;*
- (ii) $[\nabla \times \mathbf{u}^{(N-1)}] = \mathbf{0};$
- (iii) $\boldsymbol{\beta} = \mathbf{n}[\nabla \cdot \mathbf{u}^{(N-1)}].$

On the other hand, transverse waves are characterized by the next theorem.

THEOREM 4. *Given a viscoelastic wave of order N , then the following three statements are equivalent:*

- (i) *the wave is transverse;*
- (ii) $[\nabla \cdot \mathbf{u}^{(N-1)}] = 0;$
- (iii) $\boldsymbol{\beta} = [\nabla \times \mathbf{u}^{(N-1)}] \times \mathbf{n}.$

Proof of Theorems 3 and 4. By (5.3)

$$\mathbf{n} \cdot \boldsymbol{\beta} = [\nabla \cdot \mathbf{u}^{(N-1)}], \quad \mathbf{n} \times \boldsymbol{\beta} = [\nabla \times \mathbf{u}^{(N-1)}]; \quad (6.3)$$

and hence, the identity

$$\boldsymbol{\beta} = \mathbf{n}(\mathbf{n} \cdot \boldsymbol{\beta}) - \mathbf{n} \times (\mathbf{n} \times \boldsymbol{\beta}) \quad (6.4)$$

implies

$$\boldsymbol{\beta} = \mathbf{n}[\nabla \cdot \mathbf{u}^{(N-1)}] - \mathbf{n} \times [\nabla \times \mathbf{u}^{(N-1)}]. \quad (6.5)$$

Theorems 3 and 4 now follow as immediate consequences of (6.5). Equations (6.3) and (6.5) are sometimes referred to as Weingarten's First Theorem.*

7. Consequences of initial isotropy. In this section we suppress the variable \mathbf{x} and write $G_{ijkl}(t) \equiv G_{ijkl}(\mathbf{x}, t)$ and $\rho \equiv \rho(\mathbf{x})$.

We call the relaxation function $G_{ijkl}(t)$ *initially isotropic* if $G_{ijkl}(0)$ is an isotropic tensor, i.e.,

$$G_{ijkl}(0) = \frac{1}{3}(G_2 - G_1)\delta_{ij}\delta_{kl} + \frac{1}{2}G_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (7.1)$$

The constants G_1 and G_2 are respectively the *instantaneous shear modulus* and the *instantaneous modulus of pure compression*.

The solutions to the eigenvalue problem (4.1), (7.1) are well known from linear isotropic elasticity theory. Thus we have the following theorem.

THEOREM 5. *If the relaxation function is initially isotropic then there are two, and only two, possible speeds of propagation (U_L and T_T) of a viscoelastic wave. These speeds are given by:*

- (i) $U_L^2 = (G_2 + 2G_1)/(3\rho)$ in which case the wave is longitudinal;
- (ii) $U_T^2 = G_1/(2\rho)$ in which case the wave is transverse.

Conversely: if the wave is longitudinal, then $U = U_L$; if the wave is transverse, then $U = U_T$.

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*See [3], §175.

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