

ON THE HERMITE-FUJIWARA THEOREM IN STABILITY THEORY*

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1. Introduction. Let $\varphi(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial with real coefficients. In the theory of sampled-data systems (= linear, stationary, discrete-time dynamical systems, [1]), the problem of stability reduces to checking whether or not φ has any zeros on or outside the unit circle. This is a problem in algebra. The classical result is a theorem of Fujiwara [2], proved in 1926 by a method which goes back to Hermite [3] (the celebrated "Hermitian forms"):

HERMITE-FUJIWARA THEOREM. *A necessary and sufficient condition for all zeros of φ to lie inside the unit circle is that the symmetric matrix $P(\varphi)$ defined by (2.3) be positive definite.*

The same criterion was obtained slightly earlier and in slightly different form by Schur and Cohn [4].

Naturally one would like to prove the Fujiwara criterion by methods related to dynamical systems. It is well known that the second method of Lyapunov in general stability theory provides many criteria which are abstractly equivalent to that of Fujiwara and Schur-Cohn (see, e.g. [5, Sec. 7, Example 6]). In this way we can obtain an infinity of symmetric matrices $R(\varphi)$, but in contrast to $P(\varphi)$ they are rational (not integral) functions of the coefficients a_k , and therefore less convenient for practical purposes. The first concrete definition of $P(\varphi)$ via the Lyapunov theory was obtained only in 1963, by Parks [6].

The object of this note is to give a new proof, *ab initio*, of this important result of Parks. We find that there is a surprisingly close relation between the methods of Hermite and Lyapunov, and we are thereby led to a proof of the Hermite-Fujiwara theorem which is much simpler than the classical one (see, e.g., [7]).

2. Definition of $P(\varphi)$. Let $\varphi^*(z)$ be the polynomial

$$\varphi^*(z) = z^n \varphi(z^{-1}). \quad (2.1)$$

Consider the Bezoutian bilinear form [1, 8]

$$B(z, w) = \frac{\varphi(z)\varphi^*(w) - \varphi(w)\varphi^*(z)}{z - w} = \sum_{i,j=1}^n z^{i-1} \hat{p}_{ij}(\varphi) w^{j-1}. \quad (2.2)$$

By (2.1) we have also

$$B(z, w) = w^{n-1} \left[\frac{\varphi(z)\varphi(w^{-1}) - \varphi^*(w^{-1})\varphi^*(z)}{zw^{-1} - 1} \right] = w^{n-1} \left[\sum_{i,j=1}^n z^{i-1} p_{ij}(\varphi) w^{1-j} \right]. \quad (2.3)$$

Both matrices $\hat{P}(\varphi) = [\hat{p}_{ij}(\varphi)]$ and $P(\varphi) = [p_{ij}(\varphi)]$ are symmetrical. Their elements are polynomials in the a_k . Note that $p_{ij} = \hat{p}_{i, n-j}$.

*Received September 8, 1964. This research was supported in part by the US Air Force under Contract 49(638)-1206, and by the National Aeronautical and Space Administration under Contract NAS2-1107, with the Research Institute for Advanced Studies.

By comparing coefficients in (2.2) and (2.3) we obtain an explicit expression [6] for $P(\varphi)$:

$$p_{ij}(\varphi) = \sum_{k=1}^{\min(i,j)} (a_{i-k}a_{j-k} - a_{n-i+k}a_{n-j+k}); \quad (i, j = 1, \dots, n). \quad (2.4)$$

We observe that $P(\varphi)$ is a singular if φ has a zero $|z_k| = 1$. Indeed, then \bar{z}_k is also a zero of φ , $z_k^{-1} = \bar{z}_k$ is a zero of φ^* , and the bilinear form (2.4) vanishes if we let $z = z_k$, $w^{-1} = z_k^{-1} = \bar{z}_k$.

3. The lemmas. Let Φ be the companion matrix of φ ,

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \cdot & 0 & 1 & \dots & \cdot \\ \cdot & & 0 & \ddots & \cdot \\ \cdot & & & \ddots & 1 \\ -a_n & \dots & & & -a_1 \end{bmatrix}.$$

The prime will denote the transpose of a matrix or vector.

LEMMA 1 (Lyapunov-Kalman). Let r be any n -vector such that the set

$$\{(\Phi')^k r, k = 0, \dots, n-1\} = \text{linearly independent.} \quad (3.1)$$

Then all zeros of φ are inside the unit circle (i) if and (ii) only if the equation

$$\Phi' P \Phi - P = -rr' \quad (3.2)$$

has precisely one symmetric solution P and this solution is positive definite.

If the right-hand side of (3.2) is negative definite, this is a well-known result in stability theory, a counterpart of the celebrated lemma of Lyapunov for matrix differential equations with constant coefficients (see [5, 8]). Condition (3.1) is a standard device in mathematical control theory. (The pair (Φ, r') is "completely observable", see [9].)

PROOF. (i) Let x be an n -vector, and let $x(\cdot)$ be the solution of the difference equation

$$x(t+1) = \Phi x(t), \quad x(0) = x_0, \quad t = 0, 1, \dots \quad (3.3)$$

If P satisfies (3.2) and is positive definite then $x'(t)Px(t) \geq 0$ is nonincreasing as $t \rightarrow \infty$. Hence the limit exists and it must be zero because of condition (3.1). Since $x'Px = 0$ implies $x = 0$, all solutions of (3.3) converge to 0. Hence all eigenvalues of Φ (that is, all zeros of φ) are inside the unit circle.

(ii) If all eigenvalues of Φ are inside the unit circle, then the sum

$$\sum_{i=0}^{\infty} (r' \Phi^i x_0)^2 = x_0' P_1 x_0$$

exists. It is easily seen that P_1 is a solution of (3.2). And it is well-known that (3.2) has a unique solution whenever $z_1 z_2 \neq 1$ for any two zeros of φ . Finally, P_1 is positive definite by (3.1). Q.E.D.

The verification of condition (3.1) is made easier by

LEMMA 2 (Stuelpnagel). Let $[r, \dots, (\Phi')^{n-1}r] = \psi(\Phi')$, where $\psi(z) = r_1 + \dots + r_n z^{n-1}$ and r_1, \dots, r_n are the elements of r .

PROOF. See [9, Lemma 7].

Let $\pi(z)$ be the column vector with elements $1, z, \dots, z^{n-1}$.

LEMMA 3. $\Phi\pi(z) = z\pi(z) \pmod{\varphi(z)}$.

PROOF. Immediate, by computation.

Q.E.D.

Let $\psi(z) = a_n\varphi(z) - \varphi^*(z) = q_n z^{n-1} + \dots + q_1$. Let q be the column vector with elements q_1, \dots, q_n .

LEMMA 4 (Main Lemma). If $r = q(\varphi)$, the symmetric matrix $P(\varphi)$ defined by (2.3) satisfies (3.2).

PROOF. In terms of the preceding notations, we have

$$(zw^{-1} - 1)\pi'(z)P\pi(w^{-1}) = \varphi(z)\varphi(w^{-1}) - \varphi^*(z)\varphi^*(w^{-1}). \quad (3.4)$$

By Lemma 3, the left-hand side can be expressed as

$$(zw^{-1} - 1)\pi'(z)P\pi(w^{-1}) \stackrel{*}{=} \pi'(z)[\Phi'P\Phi - P]\pi(w^{-1}),$$

where $\stackrel{*}{=}$ means equality $\pmod{\varphi(z), \varphi(w^{-1})}$. On the other hand,

$$\begin{aligned} \varphi(z)\varphi(w^{-1}) - \varphi^*(z)\varphi^*(w^{-1}) &\stackrel{*}{=} -[a_n\varphi(z) - \varphi^*(z)][a_n\varphi(w^{-1}) - \varphi^*(w^{-1})], \\ &\stackrel{*}{=} -\psi(z)\psi(w^{-1}), \\ &\stackrel{*}{=} -q'\pi(z) \cdot q'\pi(w^{-1}). \end{aligned} \quad (3.5)$$

Hence

$$\pi'(z)[\Phi'P\Phi - P + qq']\pi(w^{-1}) \stackrel{*}{=} 0. \quad (3.6)$$

The highest powers of z and w^{-1} which appear on the left in (3.6) are z^{n-1} and $w^{-(n-1)}$. Hence in (3.6) we may replace $\stackrel{*}{=}$ by $=$.

For suitable choices of z , the vectors $\pi(z_1), \dots, \pi(z_n)$ are linearly independent, by Vandermonde's determinant. Similarly, we can find linearly independent vectors $\pi(w_1^{-1}), \dots, \pi(w_n^{-1})$. It follows that (3.6) implies (3.2). Q.E.D.

4. Proof of the Hermite-Fujiwara theorem. We can now prove this theorem with the Lyapunov-style Lemma 1.

Sufficiency. Suppose that P given by (2.3) is positive definite. If we can show that (3.1) holds, then it follows by Lemma 1 that all zeros of φ are inside the unit circle.

By the remark at the end of Sect. 2, φ has no zero z_k such that $|z_k| = 1$. But suppose some $|z_k| \neq 1$ is a common zero of φ and ψ . By (3.4) and (3.5) we have

$$(zw^{-1} - 1)\pi'(z)P\pi(w^{-1}) = -\psi(z)\psi(w^{-1}) \pmod{\varphi(z), \varphi(w^{-1})}.$$

If we let $z = z_k$, $w^{-1} = \bar{z}_k$, $\varphi(z_k) = \varphi(\bar{z}_k) = 0$ and P is singular, contrary to assumption. So $\psi(z_k) \neq 0$ whenever z_k is a zero of φ . By Lemma 2 and well-known facts

$$\det [q, \dots, (\Phi')^{n-1}q] = \det \psi(\Phi') = \prod_{k=1}^n \psi(z_k) \neq 0. \quad (4.1)$$

This is condition (3.1), if we take $r = q$.

Necessity. Suppose that all zeros z_k of φ are inside the unit circle. It follows that $\psi(z_k) \neq 0$ for otherwise $\varphi^*(z_k) = 0$ and $\varphi(z_k^{-1}) = 0$, which is contrary to assumption. If $q = r$, (4.1) shows that condition (3.1) is valid. Then P defined via (2.3) satisfies (3.2), by Lemma 4. In view of Lemma 1 (3.2) has exactly one symmetric solution which must be positive definite. So P is positive definite. Q.E.D.

5. New stability criterion. From the preceding proof, it follows that the Hermite-Fujiwara theorem may be replaced by the

LYAPUNOV-PARKS THEOREM. *A necessary and sufficient condition for the polynomial φ to have all its zeros inside the unit circle is that (3.2), with $r = q(\varphi)$, have a symmetric positive definite solution $P(\varphi)$. If $P(\varphi)$ exists, condition (3.1) is always valid; $P(\varphi)$ is unique and has the representation (2.3-4).*

6. Concluding remarks. As in Fujiwara [1], these results may of course be extended to provide estimates of the number of zeros on and outside the unit circle. However, this requires an *algebraic* generalization of Lemma 1. With the method of Lyapunov (theorem of Chetaev) we can prove the existence of such zeros but cannot, in general, compute their number.

We observe, without proof, that $r = q(\varphi)$ is the only vector such that (3.2) has a solution $P(\varphi)$ whose elements are integral functions of the a_k .

There is an analogous theory of polynomials whose zeros have negative real parts (see [10]).

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