LOW FREQUENCY ACOUSTIC OSCILLATIONS*

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1. Introduction. This paper concerns interior and exterior Dirichlet and Neumann problems in the plane for the equation

$$\Delta u + k^2 u = 0, \qquad (E),$$

where k is a positive constant. This is the equation governing small time periodic acoustic vibrations. The problem is to determine the behavior of solutions as k tends to zero. In the acoustic problem this corresponds to low frequency oscillations.† It is of particular interest to study the relation to the limiting problems, $k = 0$, in which case (E) becomes Laplace's equation. Many of the resulting problems can be solved explicitly using conformal mapping. Results in three dimensions can also be obtained but are not as interesting. We therefore confine ourselves to the plane.

Let Ω be a region in the plane bounded by a twice differentiable curve γ. Let Ω_c the closure of Ω and Ω_1 the complement of Ω_c . Let p denote the point (x, y), and $f(p)$ a function defined on γ. Suppose that

$$f(p) = \sum_{n=0}^{\infty} f_n(p)k^{2n},$$

the series converging uniformly on γ for $0 \leq k \leq K$ with the f_n continuous.** We write $d(p; f; k)$ and $n(p; f; k)$ for the solutions of (E) in Ω which satisfy

$$d(p; f; k) = f(p) \qquad p \in C, \tag{1.1}$$

$$n_\nu(p; f; k) = f(p), \quad \nu \text{ exterior normal}, \quad p \in C. \tag{1.2}$$

d is uniquely determined for $0 \leq k^2 < k_0^2$, k_0^2 the first eigenvalue, n is uniquely determined for $0 < k^2 < k_1^2$. We write $D(p; f; k)$ and $N(p; f; k)$ for the solutions of (E) in Ω_1 satisfying conditions (1.1) and (1.2). Uniqueness of D and N is guaranteed if we impose the radiation condition,

$$D, N \sim c(\theta)r^{-1/2} \exp (ikr) \quad \text{as} \quad r \to \infty, \tag{1.3}$$

where $r^2 = x^2 + y^2$ and $\theta = \arctan y/x$.

Consider the limit problems obtained by setting $k = 0$ while keeping conditions (1.1) and (1.2), with $f(p) = f_0(p)$. The interior Dirichlet problem has a unique solution $d(p; f_0 ; 0)$. The interior Neumann problem has no solution unless

$$\alpha(f_0) = (2\pi)^{-1} \int_\gamma f_0 \, dS = 0. \tag{1.4}$$

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†More precisely the wave length of the acoustic waves is long compared to an obstacle dimension.

**The restriction to even powers of k is made to simplify the formulas. The general case can be obtained by obvious modifications.

If (1.4) is satisfied then solutions $N(p; f_0; 0)$ exist but are only determined to within constants. The exterior Dirichlet and Neumann problems have unique solutions $D(p; f_0; 0)$ if condition (1.3) is replaced by,

$$D \text{ bounded as } r \to \infty, \tag{1.5}$$

$$N - \alpha \log r = o(1) \quad \text{as} \quad r \to \infty. \tag{1.6}$$

We summarize the results here.

Theorem 1. *There exists a K such that for $0 \leq k \leq K$,*

(i) $d(p; f; k) = \sum_{m=0}^{\infty} d_m(p; f)k^{2m}, \qquad d_0(p; f) = d(p; f_0; 0)$

(ii) $n(p; f; k) - 2\pi\alpha/Ak^2 = \sum_{m=0}^{\infty} n_m(p; f)k^{2m}, \quad A = \text{area of } \Omega,$

where $n_0(p; f) = n(p; f_0; 0)$ for a certain choice of $n(p; f_0; 0)$. The series converge uniformly in Ω.

Theorem 2. *There exists a K such that for $0 < k \leq K$*

(i) $kN(p; f; k) = \sum_{j=1}^{\infty} \sum_{i=j-1}^{\infty} N_{ij}(p; f)k^i(k \log k)^j + \sum_{i=1}^{\infty} N_i(p; f)k^i$

with

$$N_{01}(p; f) = \alpha, \qquad N_1(p; f) = \mu\alpha + N(p; f; 0),$$

where μ is a constant which is independent of f.

(ii) $D(p; f; k) = \left\{\sum_{j=0}^{\infty} \sum_{i=j}^{\infty} R_{ij}(p; f)k^{i-1}(k \log k)^{j-1}\right\} \Big/ \left\{\sum_{j=0}^{\infty} \sum_{i=j}^{\infty} S_{ij}(p; f)k^{i-1}(k \log k)^{j-1}\right\}$

with $R_{00} = R_{10} = S_{00} = S_{10} = 0$ and

$$\lim_{k \to 0} D(p; f; k) = D(p; f_0; 0).$$

The series converge uniformly, after the singular terms are deleted, in any compact subset of Ω_1.

2. Interior problems. Let u and f be continuous functions on Ω_c and γ respectively We set

$$||u|| = \max_{p \in \Omega_c} |u(p)|, \qquad ||f|| = \max_{p \in \gamma} |f(p)|.$$

We shall use the integral operators,

$$E(p; u) = -(2\pi)^{-1} \int_{\Omega}\int u(q) \log R \, dq, \ \ R = \text{distance from } p \text{ to } q$$

$$G(p; f) = -(2\pi)^{-1} \int_{\gamma} f(q) \frac{\partial}{\partial \nu_q} g(p, q) \, dS_q,$$

$$N(p; f) = -(2\pi)^{-1} \int_{\gamma} f(q) n(p, q) \, dS_q.$$

Here $g(p, q)$ and $n(p, q)$ represent respectively Green's and Neumann's functions for Laplace's equation in Ω; $n(p; q)$ is normalized by the condition

$$\frac{\partial n}{\partial \nu_q} = 2\pi/L, \qquad L = \text{length of } \gamma. \tag{2.1}$$

If u is Hölder continuous in Ω then E satisfies

$$\Delta E = u(p). \tag{2.2}$$

If f is continuous on γ then G and N represent harmonic functions in Ω which satisfy

$$G(p; f) = f(p), \qquad p \in \gamma, \tag{2.3}$$

$$N_\nu(p; f) = f(p) - \left(\int_\gamma f(q)\, dS_q\right)/L, \qquad p \in \gamma. \tag{2.4}$$

We shall need also the estimates,

$$\|E\| \leq K_1\|u\|, \qquad \|G\| \leq K_2\|f\|, \qquad \|N\| \leq K_3\|f\|. \tag{2.5}$$

We prove Theorem 1 by an explicit construction of d and n. We begin with d. Define functions $\delta_n(p)$ and $h_n(p)$ recursively by the formulas

$$\delta_0(p) = d(p; f_0; 0), \qquad h_0(p) = 0,$$

$$\delta_m(p) = -E(p; \delta_{m-1} + h_{m-1}), \qquad h_m(p) = G(p, f_m - \delta_m).$$

We set

$$d_m(p; f) = \delta_m(p) + h_m(p),$$

and have

$$\Delta d_m = -d_{m-1} \text{ in } \Omega, \qquad d_m(p; f) = f_m(p) \text{ on } \gamma.$$

Thus the series on the right side of (i) is formally a solution and it remains only to check the convergence. It can be verified that if the f_m are Hölder continuous of a common order τ the h_m will be also and so will the δ_m in Ω_c. If we apply the estimates (2.5) we find,

$$\begin{aligned} \|\delta_m\| \leq K_1(\|\delta_{m-1}\| + \|h_{m-1}\|) &\leq K_1(\|\delta_{m-1}\| + K_2\|f_{m-1}\| + K_2\|\delta_{m-1}\|) \\ &\leq M\|\delta_{m-1}\| + N\|f_{m-1}\|, \qquad m \geq 1. \end{aligned}$$

It can be verified by induction that

$$\delta_{m+1} \leq N \sum_{i=0}^{m} M^{m-i}\|f_i\| + M^{m+1}\|\delta_0\|.$$

Since the series for f converges for $0 \leq k \leq K$, we have $\|f_i\| \leq RK^{-i}$ and hence

$$\delta_{m+1} \leq NR(M + K^{-1})^m + M^{m+1}\|\delta_0\|.$$

We also have

$$h_m \leq K_2(\|f_m\| + \|\delta_m\|).$$

It follows that the series (i) converges uniformly for sufficiently small k. The derived series for the Laplacian is essentially the same and thus converges too. This completes the proof of part (*i*) of Theorem 1. The functions $n_m(p; f)$ are also defined recursively. Set

$$n_{-1} = 2\pi\alpha/A, \qquad n_m = \delta_m + h_m + c_m ,$$

where

$$\delta_{m+1} = E(p; n_m), \qquad h_m = N\left(p; f_m - \frac{\partial \delta_m}{\partial \nu}\right)$$

$$c_m = -A^{-1}\left\{- \int_\gamma f_{m+1}\, dS + \iint_\Omega (\delta_m + h_m)\, dq\right\}.$$

Then by (2.2) and (2.4)

$$\Delta n_{m+1} = n_m , \qquad \frac{\partial n_m}{\partial \nu} = \frac{\partial \delta_m}{\partial \nu} + \frac{\partial h_m}{\partial \nu} = f_m - L^{-1} \int_\gamma \left(f_m - \frac{\partial \delta_m}{\partial \nu}\right) dS.$$

Observe that

$$\int_\gamma \frac{\partial \delta_m}{\partial \nu}\, dS = \iint_\Omega \Delta \delta_m\, dq = \iint_\Omega n_{m-1}\, dq = Ac_{m-1} + \iint_\Omega (\delta_{m-1} + h_{m-1})\, dq = \int f_m\, dS,$$

hence $\partial n_m/\partial \nu = f_m$.

It follows again that (*ii*) is a formal solution, and essentially the same argument as before shows that the series converge.

3. Exterior problems. Exterior problems are most easily handled by integral equations. We begin with some results for Laplace's equation. Throughout this section p and q denote points (x, y) and (ξ, η) and R the distance between them. Consider the operators

$$A_0(p; \sigma) = (2\pi)^{-1} \int_\gamma \sigma(q) \log R\, dS_q ,$$

$$B_0(p; \sigma) = (2\pi)^{-1} \int_\gamma \sigma(q) \frac{\partial}{\partial \nu_q} \log R\, dS_q .$$

For any function σ which is continuous on γ these represent harmonic functions in Ω and in Ω_1 . They satisfy the relations

$$\lim_{p\to p_0} \left(\frac{\partial A_0}{\partial \nu}\right)^{\pm} = \mp\sigma(p_0)/2 + (2\pi)^{-1} \int_\gamma \sigma(q) \frac{\partial}{\partial \nu_p} \log R\, dS_q = S^0_{\pm}(\sigma),$$
$$\lim_{p\to p_0} (B_0)^{\pm} = \pm\sigma(p_0)/2 + (2\pi)^{-1} \int_\gamma \sigma(q) \frac{\partial}{\partial \nu_q} \log R\, dS_q = T^0_{\pm}(\sigma) \tag{4.1}$$

for $p_0 \in \gamma$. The $+$ and $-$ signs indicate limits from Ω_1 and Ω respectively. They satisfy the relations

$$A_0 - \beta \log r = O(r^{-1}), \qquad \beta = (2\pi)^{-1} \int_\gamma \sigma\, dS \quad \text{as} \quad r \to \infty ,$$
$$B_0 = O(r^{-1}) \qquad\qquad \text{as} \quad r \to \infty . \tag{4.2}$$

The use of the operators A_0 and B_0 for exterior problems is standard. We seek to express $N(p; f_0 ; 0)$ in the form

$$N(p; f_0 ; 0) = A_0(p; \sigma_0).$$

σ_0 must then satisfy the integral equation

$$S_+^0(\sigma_0) = f_0 \quad \text{on} \quad \gamma. \tag{4.3}$$

It is easy to prove, using the uniqueness theorems and (4.1) that the homogeneous equation corresponding to (4.3) has no nontrivial solutions. Thus S_+^0 can be inverted in the form

$$(S_+^0)^{-1}(f_0) = f_0(p) + \int_\gamma f_0(q)S(p, q)\, dS_q\,. \tag{4.4}$$

We note the relation

$$\beta_0 = (2\pi)^{-1} \int_\gamma \sigma_0 \, dS = (2\pi)^{-1} \int f_0 \, dS = \alpha. \tag{4.5}$$

The Dirichlet problem is more complicated. We try

$$D(p; f_0\ ; 0) = B_0(p; f_0).$$

Then σ_0 must satisfy the equation

$$T_+^0(\sigma_0) = f_0 \quad \text{on} \quad \gamma. \tag{4.6}$$

The difficulty is that the homogeneous equation has solutions. Indeed $B_0(p; l)$ is identically zero in Ω_1. It follows from (4.1) that 1 is a solution of the homogeneous equation corresponding to (4.5). It is easy to see that constants are the only solutions. We must investigate the adjoint homogeneous equation. This is

$$S_-(\sigma) = 0 \quad \text{on} \quad \gamma. \tag{4.7}$$

Let σ be a solution of (4.7) and consider $A_0(p, \sigma)$. Equation (4.7) implies that $A_0(p, \sigma)$ is a constant m in Ω. Consider then $A_0(p, \sigma)$ in Ω_1. The simple layer potential is continuous across γ hence we have $A_0(p, \sigma)^+ = m$. We map Ω_1 onto the exterior of the unit circle in the w-plane with infinity mapping into infinity. Then $A_0(p, \sigma)$ is carried into a harmonic function $A(w)$ in $|w| > 1$ which is equal to m on $|w| = 1$. Moreover (4.2) shows that

$$A(w) = \beta \log |w| + O(|w|^{-1}) \quad \text{as} \quad w \to \infty.$$

The only function of this kind is $\beta \log |w|$. Hence $m = 0$ and

$$A_0(p, \sigma) = \beta A_0(p),$$

where $A_0(p)$ is the inverse of $\log |w|$ under the mapping. We conclude from (4.1) that

$$\sigma(p) = \beta\psi(p) = \beta\left(\frac{\partial A_0(p)}{\partial \nu}\right)^+.$$

Note that

$$\tau = \int_\gamma \psi \, dS \neq 0.$$

We now apply Fredholm theory. The equation

$$T_+^0(\sigma) = h \tag{4.8}$$

can be inverted only if

$$\int_\gamma h\psi\, dS = 0 \tag{4.9}$$

If (4.8) is satisfied the general solution of (4.8) can be written as

$$\sigma(p) = h(p) + \int_\gamma h(q)T(p, q)\, dS_q + m = (T_+^0)^{-1}(h) + m, \tag{4.10}$$

where m is a constant.

We determine $D(p; f_0; 0)$ as follows. Determine M_0 by the relation

$$M_0 = \left(\int_\gamma f_0\psi\, dS\right)/\tau.$$

We can then form the expression

$$\sigma_1 = (T_+^0)^{-1}(f_0 - M_0)$$

and $B_0(p, \sigma_1) = f_0 - M_0$ on γ. Hence

$$D(p; f_0; 0) = B_0(p, \sigma_1) + M_0 .$$

We return now to equation (E). The analysis proceeds in the same way except that the logarithm must be replaced by the fundamental solution of (E) satisfying (1.3), that is,

$$G(p, q) = -(\pi/2i)H_0^{(1)}(kR)$$

where $H_0^{(1)}$ is the Hankel function of the first kind. We have the development

$$G(p, q) = \log R + \log k + G_0 + \left\{\sum_{n=1}^{\infty} (G_n(kR)^{2n} \log k + H_n(R)k^{2n}\right\}, \tag{4.11}$$

where the G_n's are constants and the series converges uniformly on compact subsets. We write $f(p, q) \sim L(k)$ for functions which have series developments like those in brackets on the right side of (4.11).

Setting

$$A(p; \sigma) = (2\pi)^{-1} \int_\gamma \sigma(q)G(p, q)\, dS_q ,$$

$$B(p; \sigma) = (2\pi)^{-1} \int_\gamma \sigma(q) \frac{\partial}{\partial \nu_q} G(p, q)\, dS_q .$$

we have

$$\begin{aligned} A(p, \sigma) &= \beta(\log k + G_0) + A_0(p, \sigma) + \int_\gamma \sigma(q)a(p, q)\, dS_q , \\ B(p, \sigma) &= B_0(p; \sigma) + \int_\gamma \sigma(q)b(p, q)\, dS_q , \end{aligned} \tag{4.12}$$

where β is as in (4.2) and $a, b \sim L(k)$, A and B satisfy (4.1) with G replacing the logarithm. We denote the corresponding boundary operators by $S_\pm$ and $T_\pm$.

We seek to express $N(p; f; k)$ in the form $A(p, \sigma)$. This yields the integral equation

$$S_+(\sigma) = f$$

for σ. Equation (4.12) shows that this may be written in the form

$$S_+^0(\sigma) + \int_\gamma \sigma(q)e(p, q)\, dS_q = f(p) = \sum_{n=0}^{\infty} f_n(p)k^{2n},$$

where $e \sim L(k)$. We invert this equation using (4.3) and interchange the order of integration to obtain

$$\sigma(p) + \int_\gamma \sigma(q)m(p, q)\, dS_q = \sum_{n=0}^{\infty} (S_+^0)^{-1}(f_n)k^{2n},$$

where $m \sim L(k)$. This equation can be solved by successive approximations. One finds

$$\sigma(p) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \sigma_{mn}k^m(k \log k)^n, \qquad \sigma_{00}(p) = (S_+^0)^{-1}(f_0) = \sigma_0 .$$

We substitute this series into $A(p; \sigma)$ to obtain $N(p; f; k)$. Thus,

$$kN(p; f; k) = \sum_{n=1}^{\infty} \sum_{n=n-1}^{\infty} N_{mn}k^m(k \log k)^n + \sum_{n=1}^{\infty} N_m k^m,$$

where

$$N_{01} = \int_\gamma \sigma_{00}\, dS = \int_\gamma \sigma_0\, dS = \int_\gamma f_0\, dS, \qquad N_1 = G_0 \int_\gamma f_0\, dS + N(p; f_0, 0).$$

This proves (*i*) of Theorem 2.

We determine $D(p; f; k)$ by a more involved procedure. By (4.1) and (4.11) we have,

$$T_+(\sigma) = T_+^0(\sigma) + \int_\gamma \sigma(q)k(p, q)\, dS_q, \tag{4.13}$$

where $k(p, q) \sim L(k)$. We set

$$J(\sigma) = \left[\int_\gamma \psi(p)\left(\int_\gamma \sigma(q)k(p, q)\, dS_q\right) dS_p\right] \Big/ \left(\int_\gamma \psi(p)\, dS_p\right).$$

Consider the equation

$$T_+^0(\sigma) = -\int_\gamma \sigma(q)k(p, q)\, dS_q + f(p) - M + J(\sigma), \tag{4.14}$$

where

$$M = \left(\int_\gamma f\psi\, dS\right) \Big/ \left(\int_\gamma \psi\, dS\right).$$

Suppose this equation has a solution σ. Then since the right side is orthogonal to ψ we could invert T_+^0 as in (4.8) and infer that σ also satisfies the equation

$$\sigma(p) = \int_\gamma \sigma(q)K(p, q)\, dS_q + (T_+^0)^{-1}(f - M) + m, \tag{4.15}$$

where $K \sim L(k)$.

We proceed as follows. Equation (4.15) can be solved, by successive approximations, for any choice of f and m. We let σ^1 be the solution when f is set equal to zero and m equals one. σ^2 will be the solution if $m = 0$. We let $u_i = B(p, \sigma^i)$. Then by construction

$$u_1(p) = J(\sigma^1), \qquad u_2(p) = f - M + J(\sigma^2) \quad \text{on} \quad \gamma.$$

We assert that $J(\sigma_1) \neq 0$ for $k > 0$ and sufficiently small. Indeed, if it were zero, u_1 would vanish identically in Ω_1. Hence

$$\left(\frac{\partial u_1}{\partial \nu}\right)^+ = \left(\frac{\partial u_1}{\partial \nu}\right)^- = 0.$$

If we require that k^2 be less than the first non-zero eigenvalue for vanishing derivative, u_1 would have to vanish in Ω_1 and hence by (4.1) σ^1 would be zero, a contradiction. It follows that

$$D(p; f; k) = u_2(p) + (M - J(\sigma^2))u_1(p)/J(\sigma^1). \tag{4.16}$$

The solution of (4.14) by successive approximations yields as before

$$\begin{aligned} \sigma^i(p) &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \sigma^i_{mn} k^m (k \log k)^n, \qquad (i = 1, 2) \\ \sigma^1_{00} &= 1\sigma^2_{00} = (T^+_0)^{-1}(f_0 - M_0) = \sigma_1, \end{aligned} \tag{4.17}$$

where it is to be recalled that $\sigma_1 = (T^0_+)^{-1}(f_0 - M_0)$. We substitute these series in $J(\sigma)$ and $B(p, \sigma)$ and obtain,

$$J(\sigma^i) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} b^i_{mn} k^m (k \log k)^n, \qquad b^i_{00} = b^i_{10} = 0,$$

$$u_i = \sum_{m=0}^{\infty} \sum_{m=n}^{\infty} a^i_{mn} k^m (k \log k)^n, \qquad a^2_{00} = B_0(p, \sigma_1).$$

We have also

$$M = \sum_{m=0}^{\infty} M_m k^m,$$

where M_0 has the same meaning as before.

We need some more detailed information concerning the various coefficients. Let γ be described by $x = x(S)$ and $y = y(S)$. Then for $p = (x(S)y(S))$ $q = (x(\sigma), y(\sigma))$ we have from (4.11)

$$\frac{\partial G}{\partial \nu_q} = \frac{\partial}{\partial \nu_q} \log R - 2G_1[(x(S) - x(\sigma))y^{\cdot}(\sigma) - (y(S) - y(\sigma))x^{\cdot}(\sigma)]k^2 \log k + o(k^2 \log k).$$

It follows that

$$\begin{aligned} \int \sigma^1(q)k(p, q)\, dS_q &= \left\{G_1\pi^{-1} \int_\gamma (x(\sigma)y^{\cdot}(\sigma) - y(\sigma)x^{\cdot}(\sigma))\, d\sigma + o(1)\right\}k^2 \log k \\ &= 2G_1\pi^{-1}Ak^2 \log k + o(k^2 \log k),\ J(\sigma_1) = 2G_1\pi^{-1}Ak^2 \log k + o(k^2 \log k). \end{aligned} \tag{4.18}$$

Now σ_1 must be a solution of (4.14) with $f = M = 0$. If we substitute the series (4.17) and use the estimates (4.18), we infer that $T^0_+(\sigma^1) = o(k^2 \log k)$, so that $T^0_+(\sigma^1_{11}) = 0$

or $\sigma_{11}^1 = \tau$, a constant. We have pointed out earlier that,

$$\int_\gamma \frac{\partial}{\partial \nu_q} \log R \, dS_q = 0.$$

Hence we deduce that

$$u_1(p) = (2\pi)^{-1} \int_\gamma \left\{ (1 + \tau k^2 \log k) \left(\frac{\partial}{\partial \nu_q} \log R - 2G_1[(x - x(\sigma))y^{\cdot}(\sigma) \right.\right.$$
$$\left.\left. - (y - y(\sigma))x^{\cdot}(\sigma)]k^2 \log k \right) \right\} d\sigma + o(k^2 \log k) = 2G\pi^{-1} A k^2 \log k + o(k^2 \log k).$$

Thus we have shown that

$$b_{11}^1 = a_{11}^1 , \tag{4.19}$$

and

$$\lim_{k \to 0} u_1(p)/J(\sigma_1) = 1.$$

We now substitute our various estimates into equation (4.16). We have

$$D(p; f; k) = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_{mn}^2 k^m (k \log k)^n + \left\{ \sum_{m=0}^{\infty} M_m k^m - \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} b_{mn}^2 k^m (k \log k)^n \right\}$$
$$\cdot \left\{ \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} a_{mn}^1 k^{m-1} (k \log k)^{n-1} \right\} \Big/ \left\{ \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} b_{mn}^1 k^{m-1} (k \log k)^{n-1} \right\}$$
$$= \left\{ \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} R_{ij} k^{i-1} (k \log k)^{j-1} \right\} \Big/ \left\{ \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} S_{ij} k^{i-1} (k \log k)^{j-1} \right\},$$

where $R_{00} = R_{10} = S_{00} = S_{10} = 0$ and

$$R_{11}/S_{11} = \lim_{k \to 0} D(p; f; k) = a_{00}^2 + M_0 - b_{00}^2 = B_0(p, \sigma_1) + M_0 = D(p; f_0 ; 0).$$

This concludes the proof of Theorem 2.

Remark. Suppose Ω is the unit circle and r and γ are polar coordinates. Then the problems we have discussed can be solved explicitly. When $f \equiv 1$ we have

$$d(p; 1; k) = J_0(kr)/J_0(k); n(p; 1; k) = -J_0(kr)/hJ_1(k),$$
$$D(p; 1; k) = H_0^{(1)}(kr)/H_0^{(1)}(k); N(p; 1; k) = -H_0^{(1)}(kr)/kH_1^{(1)}(k),$$

where the J are regular Bessel functions and the H Hankel functions. These examples illustrate that the logarithmic terms are really present in the solutions of the exterior problems.