A CLASS OF REDUCIBLE SYSTEMS OF QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. Quasi-linear partial differential equations, occurring frequently in engineering problems, are often difficult to solve. This note presents a class of systems of quasi-linear equations reducible to a single linear heat equation, and gives an example of viscous fluid flow.

2. Reduction to a linear equation. The system of n equations under consideration is of the form

$$\frac{\partial u_i}{\partial t} + F_i \frac{\partial u_i}{\partial x_j} = G_i \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j} + k \frac{\partial^2 u}{\partial x_j \partial x_j} + H_i R_i , \qquad (i, j = 1, \cdots , n) \tag{1}$$

where summation convention is adopted with index i *not* summed; F_i, G_i, and H_i are functions of u_i at least twice continuously differentiable; k is a constant and R_i a continuously differentiable function of $t, x_1 , \cdots , x_n$. With some restrictions on F_i , G_i , H_i , and R_i , Eq. (1) can be reduced to a heat equation in n dimensions.

Consider the transformation

$$F_i(u_i) = -\frac{2k}{\phi} \frac{\partial \phi}{\partial x_i}, \tag{2}$$

corresponding to which the following relations are true:

$$F_i' \frac{\partial u_i}{\partial t} = \frac{2k}{\phi^2} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x_i} - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial t \, \partial x_i},$$

$$F_i' \frac{\partial u_i}{\partial x_j} = \frac{2k}{\phi^2} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \frac{2k}{\phi} \frac{\partial^2 \phi}{\partial x_j \, \partial x_i},$$

$$\begin{aligned} F_i' \frac{\partial^2 u_i}{\partial x_j \, \partial x_j} = &-\frac{4k}{\phi^3} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} + \frac{2k}{\phi^2} \frac{\partial^2 \phi}{\partial x_j \, \partial x_j} \frac{\partial \phi}{\partial x_i} + \frac{4k}{\phi^2} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_j \, \partial x_i} \\ &- \frac{2k}{\phi} \frac{\partial^3 \phi}{\partial x_j \, \partial x_j \, \partial x_i} - \frac{4k^2}{\phi^2} \frac{F_i''}{F_i'^2} \left(\frac{1}{\phi} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_j \, \partial x_i}\right)\left(\frac{1}{\phi} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_j \, \partial x_i}\right), \end{aligned} \tag{3}$$

where prime denotes differentiation with respect to u_i and index i is not summed. Substitution of (3) into (1) yields, with i not summed,

$$\begin{aligned} \frac{2k}{\phi^2} \left(\frac{\partial \phi}{\partial x_i} - \phi \frac{\partial}{\partial x_i}\right)\left(\frac{\partial \phi}{\partial t} - k \frac{\partial^2 \phi}{\partial x_j \, \partial x_j}\right) = &F_i' H_i R_i \\ &+ \frac{4k^2}{\phi^2 F_i'^2} (G_i F_i' - k F_i'') \left(\frac{1}{\phi} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_j \, \partial x_i}\right)\left(\frac{1}{\phi} \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_i} - \frac{\partial^2 \phi}{\partial x_j \, \partial x_i}\right). \end{aligned} \tag{4}$$

Noting that the left-hand side of (4) is the partial derivative of a function with respect to x_i , we set

$$G_i F_i' - k F_i'' = 0, \quad F_i' H_i = 1, \quad R_i = -\partial P / \partial x_i , \tag{5}$$

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where i is not summed, and rewrite (4) as

$$\frac{\partial}{\partial x_i}\left[\frac{2k}{\phi}\left(\frac{\partial \phi}{\partial t} - k\,\frac{\partial^2 \phi}{\partial x_j\,\partial x_j}\right) - P\right] = 0,$$

which can be integrated to give

$$\frac{\partial \phi}{\partial t} = k\,\frac{\partial^2 \phi}{\partial x_j\,\partial x_j} + \left[c(t) + \frac{P}{2k}\right]\phi, \tag{6}$$

a linear heat equation with linear heat generation. We have shown that solutions of a system of n quasi-linear equations (1) can be obtained from the solutions of a linear equation (6). It is noted, however, that (6) will not yield all solutions of (1) because of the limitation imposed on F_i by the transformation(2).

Thus, the system of equations (1) can be reduced to a single linear equation whenever (5), which may be termed the "reducibility conditions," is satisfied. A necessary and sufficient condition for the first two of (5) is that F_i , G_i , and H_i are derived from a generating function $f_i(u_i)$ by the formulae

$$\begin{aligned} F_i &= \int^{u_i} f_i(u)\,du, \\ G_i &= kd(\ln f_i)/\,du_i\,, \\ H_i &= f_i^{-1}\,, \end{aligned} \tag{7}$$

where i is not summed. A necessary and sufficient condition for the last of (5) is that the Strokes tensor S for R_i vanishes identically

$$S_{ij} = \frac{\partial R_j}{\partial x_i} - \frac{\partial R_i}{\partial x_j} = 0. \tag{8}$$

3. Navier-Stokes equations. As an example, let us consider the Navier-Stokes equations for incompressible fluid flow

$$\frac{\partial u_i}{\partial t} + u_j\,\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x_i} + \nu\,\frac{\partial^2 u_i}{\partial x_j\,\partial x_j}, \qquad i, j = 1, 2, 3, \tag{9}$$

where u_i is the velocity component in the x_i direction, p the pressure, ρ the density, and ν the kinematic viscosity, which is assumed constant. It is easily checked that the reducibility conditions (5) are satisfied; hence through the transformation

$$u_i = -\frac{2\nu}{\theta}\,\frac{\partial \theta}{\partial x_i}, \tag{10}$$

Eq. (9) reduces to a linear heat equation

$$\frac{\partial \theta}{\partial t} = \nu\,\frac{\partial^2 \theta}{\partial x_j\,\partial x_j} + \frac{p(t, x_1\,, x_2\,, x_3)}{2\rho\nu}\,\theta. \tag{11}$$

It should be noted that the Navier-Stokes equation with *no* pressure gradient was reduced to the heat conduction equation by Cole [1]. The one-dimensional case without the pressure term was studied by Burgers [2] and Cole [1].

It is permissible to view (11) as describing a mathematical model of some viscous flow and to solve (11) as an initial value problem in infinite space with prescribed pressure $p(t_1, x_1, x_2, x_3)$; then the velocity field so obtained will need a corresponding source distribution as given by

$$Q(t, x_1, x_2, x_3) = -2\nu \frac{\partial^2 \ln \theta}{\partial x_i \, \partial x_i}$$

to satisfy conservation of mass. On the other hand, when the source distribution is specified (this case being more physical), for instance $Q \equiv 0$, Eq. (11) may be transformed into a Bernoulli's equation through elimination of $\nabla^2\theta$. This result is not surprising since in combination with the continuity equation the viscous term in (9) becomes $\nu\nabla \times (\nabla \times \mathbf{u})$ that drops off under the assumption of irrotationality implied by (10). Conversely, the nonlinear Bernoulli's equation for inviscid flow

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} + \frac{p}{\rho} = 0$$

may be converted into a linear heat equation similar to (11) by means of the equation of continuity and a change of variable $\phi = \ln \theta$.

4. Some reducible equations. A few simple forms of (1) will be listed for reference. For simplicity of presentation, only one-dimensional equations are given. Corresponding to the generating functions $f = 0, 1, e^u, nu^{n-1}, \ln u, -\sin u$, and $\cos u$ in (7), the following equations belong to the reducible class (1):

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = k \frac{\partial^2 u}{\partial x^2} + R(t, x),$$

$$\frac{\partial u}{\partial t} + e^u \frac{\partial u}{\partial x} = k\left(\frac{\partial u}{\partial x}\right)^2 + k \frac{\partial^2 u}{\partial x^2} + e^{-u}R(t, x),$$

$$\frac{\partial u}{\partial t} + u^n \frac{\partial u}{\partial x} = \frac{kn(n-1)}{u}\left(\frac{\partial u}{\partial x}\right)^2 + k \frac{\partial^2 u}{\partial x^2} + \frac{R(t, x)}{nu^{n-1}},$$

$$\frac{\partial u}{\partial t} + u(\ln u - 1) \frac{\partial u}{\partial x} = \frac{k}{u \ln u}\left(\frac{\partial u}{\partial x}\right)^2 + k \frac{\partial^2 u}{\partial x^2} + \frac{R(t, x)}{\ln u},$$

$$\frac{\partial u}{\partial t} + \begin{pmatrix}\cos u \\ \sin u\end{pmatrix} \frac{\partial u}{\partial x} = -k\begin{pmatrix}\cot u \\ \tan u\end{pmatrix}\left(\frac{\partial u}{\partial x}\right)^2 + k \frac{\partial^2 u}{\partial x^2} + \begin{pmatrix}-\csc u \\ \sec u\end{pmatrix} R(t, x),$$

where $R(t, x)$ is any function integrable with respect to x. It may be noted that in the one-dimensional case transformation (2) imposes no restriction on u more than the requirement for existence of solutions; hence every solution of the original equation may be obtained from the corresponding heat equation. Likewise, the n-dimensional systems may be derived from n generating functions, some or all of which may be identical. The one-dimensional equations listed above may provide a good visualization of the n-dimensional systems.

References

[1] J. D. Cole, *On a Quasi-Linear Parabolic Equation Occurring in Aerodynamics*, Quart. Appl. Math. **9**, 225–236 (1951)

[2] J. M. Burgers, *A Mathematical Model Illustrating the Theory of Turbulence*, Adv. Appl. Mech., Vol. **I**, Academic Press, New York, 1948, p. 171