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NONAXISYMMETRIC PUNCH AND CRACK PROBLEMS FOR INITIALLY STRESSED BODIES*

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Abstract. Using the theory developed by England and Green [1] for thermoelastic problems for initially stressed bodies and a certain class of potential functions, a group of punch and crack problems are solved. The requirement for solution is that the boundary data be expressible as a trignometric series with the coefficient of each term of the series a function of radial distance from the center of the punch or crack. The solutions are obtained by inversion of Abel's integral equation.

1. Introduction. In this paper two problems are considered. The first is concerned with the indentation of a nonsymmetrical die, having an arbitrary, small temperature distribution, into an initially stressed half-space. The second problem is concerned with the simultaneous opening and heating of a penny-shaped crack by a nonsymmetrical pressure and temperature distribution. It is assumed that all boundary data are given in the following series form:

$$\sum_{n=0}^{\infty} f_n(r) \cos (n\theta + \beta_n), \tag{1.1}$$

where (r, θ, z) are cylindrical coordinates. The coordinate system is located in the center of the disk, $(z = 0, 0 \leq r \leq a)$, where a is either the radius of the die or of the crack. The disk will therefore, define either the contact region for the punch and half-space or the location of the crack. The axis of the punch coincides with the z-axis and defines the center for the contact region of the indented half-space. The crack is in an infinite region $-\infty < z < \infty$. The solution to each problem will depend upon unknown functions that are obtained by one inversion of Abel's integral equation.

The theory to be used in this paper is based upon the recent work by England and Green [1], which is an extension of the work by Green, Rivlin, and Shield [2], and Green and Zerna [3]. For this work we consider an ideally elastic body, in a state of zero stress, strain and uniform temperature distribution. We consider the special case of an initial large, homogeneous deformation at constant temperature when two extension ratios parallel to two rectangular Cartesian coordinate axes are equal. The unequal extension ratio will be in the z-direction. The representation of displacement used here is that given by England and Green in terms of three stress functions and their results are summarized below. England and Green actually used four functions but in this paper the fourth is not needed. Only the compressible case is used in this paper since in-

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compressible problems are mathematically treated in exactly the same way. Following England and Green, the displacements are given as

$$v_i(x, y, z) + \epsilon w_i(x, y, z), \qquad (i = 1, 2, 3)$$

where (x, y, z) represent the deformed coordinates, v_i are the displacements corresponding to the finite homogeneous deformation and w_i are the infinitesimal deformations superimposed upon v_i. These are denoted by $w_1 = u$, $w_2 = v$, and $w_3 = w$. Similarly the stress tensor is $\tau^{ii} + \epsilon\tau'^{ii}$, where $\epsilon\tau'^{ii}$ is the stress corresponding to v_i. The temperature is $T + \epsilon T'$, where $\epsilon T'$ represents the additional small temperature distribution superimposed upon T. The displacement solutions to the equations of equilibrium in terms of three stress functions are as follows:

$$\begin{aligned} u &= \frac{\partial \chi_1}{\partial x} + \frac{\partial \chi_2}{\partial x} + l\,\frac{\partial \chi}{zx}, \\ v &= \frac{\partial \chi_1}{\partial y} + \frac{\partial \chi_2}{\partial y} + l\,\frac{\partial \chi}{\partial y}, \\ w &= k_1\,\frac{\partial \chi_1}{\partial z} + k_2\,\frac{\partial \chi_2}{\partial z} + m\,\frac{\partial \chi}{\partial z}, \end{aligned} \tag{1.2}$$

The corresponding equations for the stresses, τ'^{ii}, are as follows:

$$\begin{aligned} \tau'^{13} &= c_{44}\,\frac{\partial}{\partial x}\left[(1 + k_1)\,\frac{\partial \chi_1}{\partial z} + (1 + k_2)\,\frac{\partial \chi_2}{\partial z} + (l + m)\,\frac{\partial \chi}{\partial z}\right], \\ \tau'^{23} &= c_{44}\,\frac{\partial}{\partial y}\left[(1 + k_1)\,\frac{\partial \chi_1}{\partial z} + (1 + k_2)\,\frac{\partial \chi_2}{\partial z} + (l + m)\,\frac{\partial \chi}{\partial z}\right], \\ \tau'^{33} &= (c_{33}k_1 - c_{13}\nu_1)\,\frac{\partial^2 \chi_1}{\partial z^2} + (c_{33}k_2 - c_{13}\nu_2)\,\frac{\partial^2 \chi_2}{\partial z^2} + \left(c_{33}m - c_{31}\,l\,\frac{r_3}{r_1}\right)\frac{\partial^2 \chi}{\partial z^2} + \omega_3 T', \end{aligned} \tag{1.3}$$

where c_{ij} $(i, j = 1, 2, 3)$, l, m, ν_α $(\alpha = 1, 2)$, r_1, r_3, k_1, k_2, and ω_3 are all constants defined in reference one and which depend upon the three invariants of strain and upon temperature. The three unknown functions must satisfy the following equations:

$$\begin{aligned} &\nabla_1^2\chi_\alpha + \nu_\alpha\,\frac{\partial^2 \chi_\alpha}{\partial z^2} = 0, \qquad (\alpha = 1, 2) \\ &\nabla_1^2\chi + \frac{r_3}{r_1}\,\frac{\partial^2 \chi}{\partial z^2} = 0, \qquad \frac{\partial^2 \chi}{\partial z^2} = T', \\ &\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \end{aligned} \tag{1.4}$$

Here cartesian coordinates (x, y, z) are used with the z-axis corresponding to the axis of the cylindrical coordinate system (r, θ, z) mentioned earlier. In the sequel the representation for stresses and displacements as given by (1.2) to (1.4) will be used in conjunction with appropriate boundary conditions for the determination of the functions χ_1, χ_2, and χ. As a final development for this section, certain solutions to Laplace's equation will be given.

Consider the following boundary value problems defined for a half-space $0 \leq z < \infty$:

$$\begin{aligned}
\Phi^1 &= f_n(r) \cos (n\theta + \beta_n) \qquad z = 0 \qquad (0 \le r \le a) \\
\frac{\partial \Phi^1}{\partial z} &= 0, \qquad z = 0 \qquad (a < r < \infty) \\
\nabla^2 \Phi^1 &= 0 \qquad (0 < z < \infty)
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
\frac{\partial^2 \Phi^2}{\partial z^2} (r, \theta, z)\Big|_{z=0} &= g_n(r) \cos (n\theta + \beta_n), \qquad (0 \le r \le a) \\
\frac{\partial \Phi^2}{\partial z} (r, \theta, 0) &= 0, \qquad (a < r < \infty) \\
\nabla^2 \Phi^2 &= 0, \qquad (0 < z < \infty)
\end{aligned} \tag{1.6}$$

where a is the radius of a disk on the half-space, $0 \le z < \infty$, and (r, θ, z) is a cylindrical coordinate system whose center is at the center of the circle and the z-axis is perpendicular to the plane of the circle. For convenience two functions of r and z only are defined as follows:

$$\begin{aligned}
\Phi^\alpha(r, \theta, z) &= \Phi^\alpha_n(r, z) \cos (n\theta + \beta_n), \qquad (\alpha = 1, 2) \\
r^2 \nabla^2_a \Phi^\alpha_n &= n^2 \Phi^\alpha_n,
\end{aligned} \tag{1.7}$$

where ∇^2_a is the axially symmetric part of the Laplace operator, ∇^2. Then, if

$$\begin{aligned}
\Phi^1_n(r, z) &= \int_0^a t^{n+1/2} k_n(t)\, dt \int_0^\infty \alpha^{1/2} J_{n-1/2}(\alpha t) J_n(\alpha r) e^{-\alpha z}\, dz, \\
\partial \Phi^2_n/\partial z(r, z) &= \int_0^a t^{n+1/2} k_n(t)\, dt \int_0^\infty \alpha^{1/2} J_{n+1/2}(\alpha t) J_n(\alpha r) e^{-\alpha z}\, dz
\end{aligned} \tag{1.8}$$

are representations for the functions, Φ^1_n and $\partial \Phi^2_n/\partial z$, the respective functions and their normal derivatives defined on the plane, $z = 0$, are given as follows [4]:

$$\begin{aligned}
\Phi^1_n(r, 0) &= \left(\frac{2}{\pi}\right)^{1/2} r^{-n} \int_0^r \frac{t^{2n} h_n(t)\, dt}{(r^2 - t^2)^{1/2}}, \qquad (0 \le r \le a) \\
\frac{\partial \Phi^1_n}{\partial z} (r, z)\Big|_{z=0} &= \left(\frac{2}{\pi}\right)^{1/2} r^{n-1} \frac{d}{dr} \int_r^a \frac{t h_n(t)\, dt}{(t^2 - r^2)^{1/2}} \qquad (0 \le r \le a) \\
&= 0, \qquad (a < r < \infty)
\end{aligned} \tag{1.9}$$

$$\begin{aligned}
\frac{\partial \Phi^2_n}{\partial z} (r, 0) &= \left(\frac{2}{\pi}\right)^{1/2} r^n \int_r^a \frac{k_n(t)\, dt}{(t^2 - r^2)^{1/2}} \qquad (0 \le r \le a) \\
&= 0, \qquad (a < r < \infty) \\
\frac{\partial^2 \Phi^2_n}{\partial z^2} (r, z)\Big|_{z=0} &= -\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r^{n+1}} \frac{d}{dr} \int_0^r \frac{t^{2n+1} k_n(t)\, dt}{(r^2 - t^2)^{1/2}} \qquad (0 \le r \le a)
\end{aligned} \tag{1.10}$$

where $h_n(t) = k_n(t) = 0$ for $a < t < \infty$. These representations were used by the author to solve certain punch and crack problems for isotropic bodies [5]. The form of equations (1.8) represent special cases of the more general functions given by Noble [6] and Copson [7]. The values for these functions within the half-space, $0 \le z < \infty$, are as follows:

$$\Phi^1_n(r, z) = \frac{2^{1/2} \Gamma(n + 1/2) r^n}{\pi \Gamma(n)} \int_0^a t^{2n} h_n(t) \Psi^1_n(t, r, z)\, dt \tag{1.11}$$

with

$$\Psi_n^1(t, r, z) = \int_{-1}^{1} (1 - u^2)^{n-1}[(z + itu)^2 + r^2]^{-n-1/2}\, du,$$

and

$$\Phi_n^2(r, z) = \frac{\Gamma(n + 1/2)r^n}{2^{1/2}\pi\Gamma(n + 1)} \int_0^a t^{2n+1}k_n(t)\Psi_n^2(t, r, z)\, dt \tag{1.12}$$

with

$$\Psi_n^2(t, r, z) = \int_{-1}^{1} (1 - u^2)^n[(z + itu)^2 + r^2]^{-n-1/2}\, du$$

Higher normal derivatives on the disk may be computed, if they exist.

Having these solutions to two classes of problems in potential theory, one can now solve the problems discussed earlier. It will be shown that by appropriate use of the functions Φ^1 and Φ^2 the problems described earlier may be solved by elementary analysis, requiring only the inversion of Abel type integral equations characterized by (1.9) and (1.10).

2. Solution for punch problems. In this section we consider the problem of a circular, rigid punch indenting a pre-stressed elastic half-space. The face of the punch may be nonsymmetrical and lightly heated with an arbitrary temperature distribution. Complete contact between the face of the punch and the elastic half-space is assumed. In the previous notation, the boundary conditions are:

$$\begin{aligned} w &= g_n(r)\cos(n\theta + \beta_n), & z &= 0, & (0 \le r \le a) \\ \tau'^{33} &= 0, & z &= 0, & (a < r < \infty) \\ \tau'^{13} &= \tau'^{23} = 0, & z &= 0, & (0 \le r < \infty) \end{aligned} \tag{2.1}$$

$$\begin{aligned} T' &= f_n(r)\cos(n\theta + \beta_n), & z &= 0, & (0 \le r \le a) \\ T' &= 0, & z &= 0, & (a < r < \infty) \end{aligned} \tag{2.2}$$

It is understood that by this formulation of displacement and temperature distribution, very general combinations of distribution may be obtained by using superposition solutions.

Boundary conditions (2.1) are satisfied if

$$(1 + k_1)\frac{\partial\chi_1}{\partial z} + (1 + k_2)\frac{\partial\chi_2}{\partial z} + (l + m)\frac{\partial\chi}{\partial z} = 0, \quad z = 0 \qquad (0 \le r < \infty) \tag{2.3}$$

$$(c_{33}k_1 - c_{31}\nu_1)\frac{\partial^2\chi_1}{\partial z^2} + (c_{33}k_2 - c_{31}\nu_2)\frac{\partial^2\chi_2}{\partial z^2} + \left(c_{33}m - c_{31}l\frac{r_3}{r_1} + \omega_3\right)\frac{\partial^2\chi}{\partial z^2} = 0, \quad z = 0 \qquad (a < r < \infty) \tag{2.4}$$

$$k_1\frac{\partial\chi_1}{\partial z} + k_2\frac{\partial\chi_2}{\partial z} + m\frac{\partial\chi}{\partial z} = g_n(r)\cos(n\theta + \beta_n), \quad z = 0 \qquad (0 \le r \le a) \tag{2.5}$$

where $\partial^2\chi/\partial z^2 = T'$ has been used. To satisfy (2.3) to (2.5), we now put

$$\chi_1 = \left\{\alpha\chi_n\left[r, z\left(\frac{r_3}{r_1\nu_1}\right)^{1/2}\right] + \frac{\nu_1^{1/2}}{1+k_1}\Phi_n\left(r, \frac{z}{\nu_1^{1/2}}\right)\right\}\cos(n\theta+\beta_n)$$
$$\chi_2 = \left\{\beta\chi_n\left[r, z\left(\frac{r_3}{r_1\nu_2}\right)^{1/2}\right] - \frac{\nu_2^{1/2}}{1+k_2}\Phi_n\left(r, \frac{z}{\nu_2^{1/2}}\right)\right\}\cos(n\theta+\beta_n) \tag{2.6}$$

and choose α and β so that (2.3) and (2.4) are satisfied identically in χ. This is the case if α and β are defined as solutions to the following pair of equations:

$$(1+k_1)\left(\frac{r_3}{r_1\nu_1}\right)^{1/2}\alpha + (1+k_2)\left(\frac{r_3}{r_1\nu_2}\right)^{1/2}\beta = -(l+m),$$
$$(c_{33}k_1 - c_{31}\nu_1)\left(\frac{r_3}{r_1\nu_1}\right)\alpha + (c_{33}k_2 - c_{31}\nu_2)\left(\frac{r_3}{r_1\nu_2}\right)\beta = -\left(c_{33}m - c_{31}\,l\frac{r_3}{r_1} + \omega_3\right). \tag{2.7}$$

The remaining boundary conditions reduce to the following:

$$\left[\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right]\frac{\partial\Phi_n}{\partial z} = -A\frac{\partial\chi_n}{\partial z} + g_n(r), \quad z = 0, \qquad (0 \le r \le a)$$
$$\frac{\partial^2\Phi_n}{\partial z^2} = 0, \qquad z = 0, \qquad (a < r < \infty)$$
$$r^2\nabla_a^2\Phi_n = n^2\Phi_n, \tag{2.8}$$

where $A = \alpha k_1(r_3/r_1\nu_1)^{1/2} + \beta k_2(r_3/r_1\nu_2)^{1/2} + m.$

To solve this set of equations $\partial\chi/\partial z$ must be determined on the plane, $z = 0$. Considering the second of equations (1.5), boundary conditions (2.2) and equations (1.8), we see that a suitable integral for T' that satisfies the second condition of equation (2.2) is

$$T' = \int_0^a t^{n+1/2}k_n^1(t)\,dt\int_0^\infty \alpha^{1/2}J_{n+1/2}(\alpha t)J_n(\alpha r)e^{-\alpha\zeta}d\alpha\cos(n\theta+\beta_n), \tag{2.9}$$

where $\zeta = z(r_1/r_3)^{1/2}$. We observe that when $\zeta = 0$, the first condition (2.2) becomes

$$\left(\frac{2}{\pi}\right)^{1/2} r^n\int_r^a \frac{k_n^1(t)\,dt}{(t^2-r^2)^{1/2}} = f_n(r), \quad (0 \le r \le a). \tag{2.10}$$

This integral equation is solved by elementary analysis to yield

$$k_n^1(t) = -\left(\frac{2}{\pi}\right)^{1/2}\frac{d}{dt}\int_t^a \frac{r^{1-n}f_n(r)\,dr}{(r^2-t^2)^{1/2}}. \tag{2.11}$$

Alternatively,

$$T' = \frac{2^{1/2}}{\pi}\frac{\Gamma(n+1)r^n}{\Gamma(n+\frac{1}{2})}\int_0^a t^{2n+1}k_n^1(t)\Psi_n^2(t, r, z)\,dt, \tag{2.12}$$

where

$$\Psi_n^2(t, r, z) = \int_{-1}^1 (1-u^2)^{n-\frac{1}{2}}[(\zeta+iru)^2 + t^2]^{-n-1}\,du.$$

Remark. To compute the particular integral for χ, one must twice integrate Eq. (2.9). The result involves $\zeta\varphi_1 + \varphi_2$ where φ_1 and φ_2 are arbitary plane harmonic functions.

This indeterminacy can be removed by imposing the condition that all stress components vanish as $R \to \infty$. The value for χ and its first derivative will not, in general, vanish which implies that components of displacement may exist at infinity.

$$\left(\frac{r_3}{r_1}\right)^{1/2} \frac{\partial \chi_n}{\partial z} = -\int_0^a t^{n+1/2} k_n^1(t)\, dt \int_0^\infty \alpha^{-1/2} J_{n+1/2}(\alpha t) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha,$$

$$\chi_n \cos(n\theta + \beta_n) = \chi, \tag{2.13}$$

or after an integration by parts

$$\left(\frac{r_1}{r_3}\right)^{1/2} \frac{\partial \chi_n}{\partial z} = -a^{n+1/2} l_n(a) \int_0^\infty \alpha^{-1/2} J_{n+1/2}(\alpha a) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha + \int_0^a t^{n+1/2} l_n(t) \int_0^\infty \alpha^{1/2} J_{n-1/2}(\alpha t) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha, \tag{2.14}$$

where $l_n(t) = \int_0^t k_n^1(t)\, ds$. On $z = 0$ this integral may be evaluated as follows:

$$\left(\frac{r_1}{r_3}\right)^{-1/2} \frac{\partial \chi_n}{\partial z} = -a^{n+1/2} l_n(a) \int_0^\infty \alpha^{-1/2} J_{n+1/2}(\alpha a) J_n(\alpha r)\, d\alpha + \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{t^{2n} l_n(t)\, dt}{(r^2 - t^2)^{1/2}}. \tag{2.15}$$

Thus the boundary cond'tion for the right hand side of the second Eq. (2.8) is known. The function, χ_n, is determined by another integration of (2.14), and its value is

$$\left(\frac{r_1}{r_3}\right) \chi_n = \int_0^a t^{n+1/2} l_n(t)\, dt \int_0^\infty \alpha^{-1/2} J_{n-1/2}(\alpha t) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha + a^{n+1/2} l_n(a) \int_0^\infty \alpha^{-3/2} J_{n+1/2}(\alpha a) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha. \tag{2.16}$$

Another representation for (2.16) is as follows:

$$\left(\frac{r_1}{r_3}\right) \chi_n = \frac{\Gamma(n) r^n}{2^{1/2} \pi \Gamma(n + \frac{1}{2})} \int_0^a t^{2n} l_n(t)\, dt \int_{-1}^1 \frac{(1 - u^2)^{n-1/2}\, du}{[(\zeta + iru)^2 + t]^n} - a^{n+1/2} l_n(a) \int_0^\infty \alpha^{-3/2} J_{n+1/2}(\alpha a) J_n(\alpha r) e^{-\alpha \zeta}\, d\alpha, \tag{2.17}$$

where the last integral of the equation (2.17) can be expressed in terms of hypergeometric functions.

The problem is solved by finding the solution to the problem posed by equations (2.8). A function that satisfies boundary conditions (2.8) is the following:

$$\frac{\partial \Phi_n}{\partial z} = \int_0^a t^{n+1/2} h_n^1(t)\, dt \int_0^\infty \alpha^{1/2} J_{n-1/2}(\alpha t) J_n(\alpha r) \exp[-\alpha z/\nu^{1/2}]. \tag{2.18}$$

On the disk, $z = 0$ $(0 \le r \le a)$, equation (2.8) becomes with the aid of equation (2.15) and (2.18).

$$\left(\frac{k_1}{1 + k_1} - \frac{k_2}{1 + k_2}\right)\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{t^{2n} h_n^2(t)\, dt}{(r^2 - t^2)^{1/2}} = \left(\frac{r_3}{r_1}\right)^{1/2} A \left[- a^{n+1/2} l_n(a) \times \int_0^\infty \alpha^{-1/2} J_{n+1/2}(\alpha a) J_n(\alpha r)\, d\alpha + \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{t^{2n} l_n(t)\, dt}{(r^2 - t^2)^{1/2}} \right] + g_n(r), \tag{2.19}$$

The value of $h_n^1(t)$ is found to be

$$\left(\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right) h_n^1(t) = \left(\frac{r_3}{r_1}\right)^{1/2} A[-l_n(a) + l_n(t)] + \left(\frac{2}{\pi}\right)^{1/2} \frac{t^{-2n}d}{dt} \int_0^t \frac{r^{n+1} g_n(r)\, dr}{(t^2 - r^2)^{1/2}}, \tag{2.20}$$

where Eq. 8.11 (1) of reference [4] and the following form of Sonine's first integral have been used:

$$t^{n+1/2} J_{n+1/2}(\alpha t) = \left(\frac{2\alpha}{\pi}\right)^{1/2} \int_0^t \frac{r^{n+1} J_n(\alpha r)\, dr}{(t^2 - r^2)^{1/2}}.$$

The value for Φ_n, obtained by integrating (2.18), is

$$\Phi_n = \left(\frac{\nu_1}{2}\right)^{1/2} \frac{\Gamma(n)}{\pi\Gamma(n+\frac{1}{2})} r^n \int_0^a t^{2n} h_n^1(t)\, dt \int_{-1}^1 \frac{(1-u^2)^{n-1/2}\, du}{[(z\nu_1^{-1/2} + iru)^2 + t^2]^n}. \tag{2.21}$$

Hence, the determination of Φ and χ is completed and the punch problem is solved. The next section will consider an analogous problem for a penny-shaped crack.

3. Solution for crack problems. In this section we consider the problem of a penny-shaped crack being opened in an infinite pre-stressed elastic medium. The crack is opened by a small prescribed normal pressure, and in addition, the faces of the crack are heated by a small nonsymmetrical temperature distribution. The work in this section will directly relate to that of section 7 in England and Green. The boundary conditions are written as follows:

$$\begin{aligned} \tau'^{33} &= g_n(r)\cos(n\theta + \beta_n), && z = 0, \quad (0 \le r \le a) \\ w &= 0, && z = 0, \quad (a < r < \infty) \\ \tau'^{13} &= \tau'^{23} = 0, && z = 0, \quad (0 \le r < \infty) \end{aligned} \tag{3.1}$$

$$\begin{aligned} T' &= f_n(r)\cos(n\theta + \beta_n), && z = 0, \quad (0 \le r \le a) \\ \frac{\partial T'}{\partial z} &= 0, && z = 0, \quad (a < r < \infty). \end{aligned} \tag{3.2}$$

Boundary conditions (3.1) are satisfied if

$$(1+k_1)\frac{\partial \chi_1}{\partial z} + (1+k_2)\frac{\partial \chi_2}{\partial z} + (l+m)\frac{\partial \chi}{\partial z} = 0, \quad z = 0, \quad (a \le r < \infty) \tag{3.3}$$

$$k_1 \frac{\partial \chi_1}{\partial z} + k_2 \frac{\partial \chi_2}{\partial z} + m\frac{\partial \chi}{\partial z} = 0, \quad z = 0, \quad (a < r < \infty) \tag{3.4}$$

$$\begin{aligned} (c_{33}k_1 - c_{31}\nu_1)\frac{\partial^2 \chi_1}{\partial z^2} + (c_{33}k_2 - c_{31}\nu_2)\frac{\partial^2 \chi_2}{\partial z^2} + (c_{33}m - c_{31}lr_3/r_1)\frac{\partial^2 \chi}{\partial z^2} + \omega_3 T' \\ = g_n(r)\cos(n\theta + \beta_n), \quad z = 0, \quad (0 \le r \le a). \end{aligned} \tag{3.5}$$

Put

$$\begin{aligned} \chi_1 &= \left\{\alpha\chi_n[r, z(r_3/r_1\nu_1)^{1/2}] + \frac{\nu_1^{1/2}}{1+k_1}\Phi_n(r, z\nu_2^{-1/2})\right\}\cos(n\theta + \beta_n), \\ \chi_2 &= \left\{\beta\chi_n[r, z(r_3/r_1\nu_2)^{1/2}] - \frac{\nu_2^{1/2}}{1+k_2}\Phi_n(r, z\nu_2^{-1/2})\right\}\cos(n\theta + \beta_n), \end{aligned} \tag{3.6}$$

where

$$\alpha(r_3/r_1\nu_1)^{1/2} = \frac{(lk_2 - m)}{k_1 - k_2}, \qquad \beta(r_3/r_1\nu_2)^{1/2} = \frac{(m - lk_1)}{k_1 - k_2}$$

$$r^2\nabla_a^2\Phi_n = n^2\Phi_n.$$

Then the remaining boundary conditions become

$$\begin{aligned} \frac{\partial^2\Phi_n}{\partial z^2} &= Xg_n(r) - Yf_n(r), \qquad z = 0, \qquad (0 \le r \le a) \\ \frac{\partial\Phi_n}{\partial z} &= 0, \qquad\qquad\qquad\quad z = 0, \qquad (a < r < \infty) \end{aligned} \tag{3.7}$$

where

$$X^{-1} = \frac{c_{33}k_1 - c_{31}\nu_1}{(1 + k_1)\nu_1^{1/2}} - \frac{c_{33}k_2 - c_{31}\nu_2}{(1 + k_2)\nu_2^{1/2}},$$

$$YX^{-1} = c_{33}m - c_{31}lr_3/r_1 + \omega_3 + (c_{33}k_1 - c_{31}\nu_1)\alpha r_3/r_1\nu_1 + (c_{33}k_2 - c_{31}\nu_2)\beta r_3/r_1\nu_2.$$

These boundary conditions are satisfied with the help of the solution to the problem given by (1.5). Hence, choose

$$\frac{\partial\Phi_n}{\partial z} = \int_0^a t^{n+1/2}h_n^2(t)\,dt \int_0^\infty \alpha^{1/2}J_{n+1/2}(\alpha t)J_n(\alpha r)\exp(-\alpha z\nu_1^{-1/2}). \tag{3.8}$$

By analogy with (1.10), the following integral equation is established:

$$-\left(\frac{2}{\pi}\right)^{1/2} r^{-n-1}\frac{d}{dr}\int_0^r \frac{t^{2n+1}h_n^2(t)\,dt}{(r^2 - t^2)^{1/2}} = Xg_n(r) - Yf_n(r), \tag{3.9}$$

where

$$h_n^2(t) = -\left(\frac{2}{\pi}\right)^{1/2} t^{-2n}\int_0^t \frac{r^{n+1}[Xg_n(r) - Yf_n(r)]\,dr}{(t^2 - r^2)^{1/2}}$$

is the solution to equation (3.9). This completes the determination for $\partial\Phi_n/\partial z$. To obtain Φ_n one must formally integrate equation (3.8) with respect to z. The integration is easily completed when equation (2.17) is written in the form given by equation (1.8). The result is

$$\nu^{-1/2}\Phi_n = \frac{2^{1/2}\Gamma(n + 1/2)r^n}{\pi\Gamma(n + 1)}\int_0^a h_n^2(t)t^{2n+1}\,dt\int_{-1}^1 \frac{(1 - u^2)\,du}{[(z\nu^{-1/2} + itu) + r^2]^{n+1/2}}. \tag{3.10}$$

Finally, the temperature problem as posed by equations (3.2) must be solved.

Considering the second Eq. (1.5), boundary conditions (3.2), and Eqs. (1.8), we see that a suitable integral for T' in this case is

$$T' = \int_0^a t^{n+1/2}k_n^2(t)\,dt\int_0^\infty \alpha^{1/2}J_{n-1/2}(\alpha t)J_n(\alpha r)e^{-\alpha\zeta}\,d\alpha\cos(n\theta + \beta_n), \tag{3.11}$$

where $\zeta = z(r_1/r_3)^{1/2}$.

When $\zeta = 0$, the first condition in equation (2.2) becomes

$$\left(\frac{2}{\pi}\right)^{1/2} r^{-n}\int_0^r \frac{t^{2n}k_n^2(t)\,dt}{(t^2 - r^2)^{1/2}} = f_n(r), \tag{3.12}$$

where

$$k_n^2(t) = \left(\frac{2}{\pi}\right)^{1/2} t^{-2n} \frac{d}{dt} \int_0^t \frac{r^{n+1} f_n(r)\, dr}{(t^2 - r^2)^{1/2}}. \tag{3.13}$$

Alternatively, the temperature may be expressed in the following way:

$$T' = \frac{2^{1/2}\Gamma(n + 1/2) r^n}{\pi \Gamma(n)} \int_0^a t^{2n} k_n^2(t) \Psi_n^1(t, r, z)\, dt, \tag{3.14}$$

where

$$\Psi_n^1(t, r, z) = \int_{-1}^1 \frac{(1 - u^2)^{n-1}\, du}{[(z + itu)^2 + r^2]^{n+1/2}}.$$

The computation of χ_n requires that (3.12) be integrated twice with respect to z. Using the same argument as before, the arbitrary constants arising from the integration may be set equal to zero. One integration gives the following result for $\partial\chi_n/\partial z$:

$$\left(\frac{r_1}{r_3}\right)^{1/2} \frac{\partial \chi_n}{\partial z} = \frac{\Gamma(n) r^n}{2^{1/2}\pi\Gamma(n + \frac{1}{2})} \int_0^a t^{2n} k_n^2(t)\, dt \int_{-1}^1 \frac{(1 - u^2)^{n-1/2}\, du}{[(\zeta + iru)^2 + t^2]^n}. \tag{3.15}$$

A further integration determines χ_n as

$$\left(\frac{r_1}{r_3}\right) \chi_n = \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(2n + 2m - 1)}{m!\Gamma(m + n + \frac{1}{2})} \cdot \int_0^a t^{n+1/2} k_n^2(t) \left(\frac{t}{2\zeta}\right)^{2m} {}_2F_1[-m, -n - m - \tfrac{1}{2}; n + 1; r^2/t^2]. \tag{3.16}$$

The values obtained for Φ_n and χ_n complete the solution to the crack problem posed by boundary conditions (3.1) and (3.2).

4. Examples. To illustrate the theory developed in the earlier sections and to describe some of the requirements that must be imposed when solving a nonsymmetrical punch or crack problem, we first consider the following punch problem:

$$\begin{aligned} \omega &= \delta + \epsilon r \cos 3\theta, && z = 0, && (0 \le r \le a) \\ \tau'^{33} &= 0, && z = 0, && (a < r < \infty) \\ \tau'^{13} &= \tau'^{23} = 0, && z = 0, && (0 \le r < \infty) \end{aligned} \tag{4.1}$$

$$\begin{aligned} T' &= K(a^2 - r^2)^{1/2}, && z = 0, && (0 \le r \le a) \\ T' &= 0, && z = 0, && (a < r < \infty). \end{aligned} \tag{4.2}$$

From (2.10) and (2.11) the axisymmetric temperature problem is solved with

$$k_0^1(t) = K\left(\frac{\pi}{2}\right)^{1/2} t. \tag{4.3}$$

From (2.15) and (4.3) the normal derivative on the disk, $z = 0$ $(0 \le r \le a)$ is found as

$$\left(\frac{r_1}{r_3}\right)^{1/2} \frac{\partial \chi}{\partial z}\bigg|_{z=0} = K\left(\frac{\pi}{2}\right)^{1/2} \left[-\frac{a^{5/2}}{2}\int_0^\infty J_{1/2}(\alpha a) J_0(\alpha r)\alpha\, d\alpha + \frac{r^2}{4}\right], \qquad (0 \le r \le a). \tag{4.4}$$

Having these values, one must now find the solution to the following boundary value problem:

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = R\left[\left(\frac{r_3}{r_1}\right)^{1/2} A\,\frac{\partial\chi}{\partial z} + \delta + \epsilon r\cos 3\theta\right], \qquad (0 \le r \le a)$$

$$\left.\frac{\partial^2\Phi}{\partial z^2}\right|_{z=0} = 0, \qquad z = 0, \qquad (a < r < \infty) \tag{4.5}$$

$$\nabla^2\Phi = 0, \qquad (0 < z < \infty)$$

where

$$R = \left[\frac{k_1}{1+k_1} - \frac{k_2}{1+k_2}\right]^{-1}$$

The solution is accomplished by direct application of (2.20) to yield

$$h_0^1 = R\left[\left(\frac{r_3}{r_1}\right)^{1/2} AK\left(\frac{\pi}{2}\right)^{1/2} \frac{1}{2}(a^2 - t^2) + \left(\frac{2}{\pi}\right)^{1/2}\delta\right], \tag{4.6}$$

$$h_3^1 = R\left(\frac{2}{\pi}\right)^{1/2} \frac{8}{3}\,\epsilon t^{-2}. \tag{4.7}$$

The problem is solved except for restrictions which must be placed on the constants K, ϵ, and δ. They must be so chosen that the face of the disk makes complete contact with the half-space on the disk, $z = 0$, $(0 \le r \le a)$.

If the contact is incomplete, then use of the results given in part two of this paper is not valid. A relation between the three constants must be found so that the condition

$$\tau'^{33} \le 0 \tag{4.8}$$

is always met. From equations (1.9), the third of (1.3), and (2.6)

$$\tau'^{33} = B\,\frac{\partial^2\phi}{\partial z^2} = -BR\left\{\frac{AK}{2}\left(\frac{r_3}{r_1}\right)^{1/2}(a^2 - r^2)^{1/2} + \frac{2}{\pi}\,\delta(a^2 - r^2)^{-1/2}\right.$$
$$\left. + \frac{16}{3\pi}\,\epsilon\left[\cos^{-1}\frac{r}{a} + r(a^2 - r^2)^{-1/2}\right]\cos 3\theta\right\}, \qquad z = 0, \qquad (0 \le r \le a) \tag{4.9}$$

where $B = (c_{33}k_1 - c_{31}\nu_1)/(1 + k_1)\nu_1^{1/2} - (c_{33}k_2 - c_{31}\nu_2)/(1 + k_2)\nu_2^{1/2}$. Conditions that ensure complete contact are

$$\frac{2}{\pi}\frac{\delta}{a} + \frac{aAK}{2}\left(\frac{r_3}{r_1}\right)^{1/2} \ge 0, \tag{4.10}$$

$$\frac{2}{\pi}\frac{\delta}{a} + \frac{aAK}{2}\left(\frac{r_3}{r_1}\right)^{1/2} - \frac{16}{3\pi}\,\epsilon\,\frac{\pi}{2} \ge 0. \tag{4.11}$$

Hence, if (4.11) is satisfied then there will certainly be complete contact between die and half space.

The physical restriction to be satisfied for a crack problem is that the two sides of the crack do not overlap when the crack is simultaneously heated and stressed. A problem similar to the preceding will illustrate the method of solution. The boundary conditions are as follows:

$$\tau'^{33} = -\epsilon, \qquad z = 0, \qquad (0 \le r \le a)$$

$$w = 0, \qquad z = 0, \qquad (0 < r < \infty) \tag{4.12}$$

$$\tau'^{13} = \tau'^{23} = 0, \qquad z = 0, \qquad (0 \le r < \infty)$$

$$T' = \delta r \cos 3\theta, \qquad z = 0, \qquad (0 \le r \le a)$$
$$\frac{\partial T'}{\partial z} = 0, \qquad z = 0, \qquad (a < r < \infty). \tag{4.13}$$

The function $\partial\Phi_n/\partial z$ defined by (3.8) is determined once $h_0^2(t)$ and $h_3^2(t)$ have been found. Using boundary conditions (4.12) and (4.13) with (3.10) the required functions are found to have the following values:

$$h_0^2(t) = \left(\frac{2}{\pi}\right)^{1/2} \epsilon X \int_0^t r(t^2 - r^2)^{-1/2}\, dr = \left(\frac{2}{\pi}\right)^{1/2} \epsilon X t, \tag{4.14}$$

$$h_3^2(t) = -\left(\frac{2}{\pi}\right)^{1/2} \delta Y t^{-6} \int_0^t r^5(t^2 - r^2)^{-1/2}\, dr = -\left(\frac{2}{\pi}\right)^{1/2} \delta Y \frac{8}{15} t^{-1}. \tag{4.15}$$

To ensure that the displacements of the crack do not overlap one must compute their value on the disk, $z = 0$, $(0 \le r \le a)$. From (4.14), (4.15) and (3.6) the displacement may be computed as

$$R^{-1}w = \frac{2}{\pi} \epsilon X(a^2 - r^2)^{1/2} - \frac{16}{15\pi} \delta Y r^2 \cos^{-1}\left(\frac{r}{a}\right) \cos 3\theta. \tag{4.16}$$

To ensure that the faces of the crack do not meet the following inequality must hold:

$$w \ge 0, \qquad z = 0, \qquad (0 \le r \le a). \tag{4.17}$$

The inequality (4.17) is ensured if ϵ and δ are related by

$$\epsilon \ge 8\delta Y a/15X. \tag{4.18}$$

The solution is completed when $k_3^2(t)$ is determined. From equation (3.13)

$$k_3^2(t) = (2/\pi)^{1/2} 8\delta/3t^2. \tag{4.19}$$

5. Conclusion. Using the results of certain Hankel transforms and the theory developed by England and Green, one can obtain solutions to nonsymmetrical punch and crack problems with nonsymmetrical temperature distributions for initially stressed elastic bodies. The solutions will involve finite elementary integrals and may be obtained by inversion of Abel's integral equation.

Further nonsymmetrical problems that can be worked by the same methods are steady state thermoelastic problems for transversely isotropic and anisotropic materials. The analysis for both punch and crack problems will proceed in exactly the same way as described in this paper.

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