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ON OSCILLATION NUMBERS OF SECOND ORDER LINEAR DIFFERENTIAL SYSTEMS*

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1. The purpose of this paper is to obtain sufficient conditions for the existence of a non-oscillatory solution $x(t)$ of

$$x'' + F(t)x = 0, \tag{1.1}$$

where $x = (x_1, x_2, \cdots, x_m)$, and $F = F(t)$ is an m by m matrix which is a continuous function of t. Since m may be replaced by $2m$, there is no loss of generality in assuming that the m^2 elements of F are real-valued. Thus, the m components of the vector x on which F operates will be confined to the real field.

Consider only those solutions $x = x(t)$ of (1.1) which are different from $x(t) \equiv 0$. We use A. Wintner's [2] definition of the oscillation number F_θ of (1.1) to be the least value having the property that no solution vector $x(t) \not\equiv 0$ will become the zero vector at more than F_θ points t of θ, any t-interval of finite or infinite length and we will also require that each component solution $x_i(t)$ be simultaneously positive or negative between consecutive zeros on θ.

Let $x \cdot y$ denote the scalar product $\sum_{k=1}^{m} x_k y_k$ of the vectors $x = (x_1, x_2, \cdots, x_m)$ and $y = (y_1, y_2, \cdots, y_m)$.

2. Non-oscillatory solutions of (1.1). We have the following results.

THEOREM. Let the following conditions be satisfied:

(*i*) $F(t)$ is a real symmetric positive definite m by m matrix for $t \geq 0$;

(*ii*) $F(t)$ is non-increasing, i.e., if $t > s$, then $F(t) - F(s)$ is non-positive;

(*iii*) The determinant of $F(t)$ tends to zero as t tends to ∞;

(*iv*) $\int_0^\infty t \max_{i,j} [a(t)]_{ij} \, dt < \infty, \; F(t) = (a_{ij}), \qquad a_{ij} \geq 0.$

Then (1.1) is non oscillatory.

P. Hartman [1] has shown that (*i*), (*ii*), (*iii*) imply that (1.1) has a solution $x(t)$ for which

$$F(t)x \cdot x + x' \cdot x' \to 0, \quad \text{as} \quad t \to \infty. \tag{1.2}$$

This implies that for $x(t)$ a solution of (1.1), $x'(t)$ remains bounded as $t \to \infty$. Let us assume that (1.1) has a solution $x(t)$ for which (1.2) holds, and has an oscillation number F_θ which is unbounded. Let

$$\{t_1, t_2, t_3, \cdots, t_\lambda, t_{\lambda+1}, \cdots\} \tag{1.3}$$

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be successive zeros of $x(t)$ and let $x_i'(t) > 0$. Now $x_i'(t)$ has a unique smallest zero t_λ^* for t in $t_\lambda < t < t_{\lambda+1}$.

Integrating (1.1) component-wise over (t_λ, t_λ^*) we get

$$x_i'(t_\lambda^*) - x'(t_\lambda) + \int_{t_\lambda}^{t_\lambda^*} \sum_{i,j=1}^{m} a_{ij}(t)x_j(t)\,dt = 0, \tag{1.4}$$

or,

$$x_i'(t) = \int_{t_\lambda}^{t_\lambda^*} \sum_{i,j=1}^{m} a_{ij}(t)x_j(t)\,dt. \tag{1.5}$$

Now $x_i(t_\lambda) = 0$, for $(i = 1, 2, \cdots, m)$, and $x_i'(t_\lambda)$ is either positive and decreasing, or negative and increasing in (t_λ, t_λ^*). Thus for $t_\lambda \le t \le t_\lambda^*$, we have,

$$0 \le |x_i(t)| \le |x_i'(t_\lambda)| \cdot |t - t_\lambda|, (i = 1, 2, \cdots, m). \tag{1.6}$$

Making use of (1.5), (1.6) we get,

$$(|x_1'(t_\lambda)| + \cdots + |x_m'(t_\lambda)|) \le (|x_1'(t_\lambda)| + \cdots + |x_m'(t_\lambda)|) \int_{t_\lambda}^{t_\lambda^*} tm \max_{i,j} [a_{ij}(t)]\,dt \tag{1.7}$$

Thus,

$$1 \le \int_{t_\lambda}^{t_\lambda^*} tm \max_{i,j} [a_{ij}(t)]\,dt. \tag{1.8}$$

This is impossible as $t \to \infty$, since $|x'(t_\lambda)|$, for $(i = 1, 2, \cdots, m)$, remains bounded as $t \to \infty$, and by (iv) we have,

$$\int_{t_\lambda}^{\infty} t \max_{i,j} [a_{ij}(t)]\,dt \to 0, \quad \text{as} \quad t \to \infty. \tag{1.9}$$

So $F_\theta < \infty$, and (1.1) has a solution $x(t)$ which is non-oscillatory.

References

1. P. Hartman, *The existence of large or small solutions of linear defferential equations*, Duke Math. J. **38** (1961) 421
2. A. Wintner, *A comparison theorem for Sturmian oscillation numbers of linear systems of second order*, Duke Math. J., **25** (1958) 515