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A UNIQUENESS THEOREM FOR THE DISPLACEMENT BOUNDARY-VALUE PROBLEM OF LINEAR ELASTODYNAMICS*

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1. Introduction. The displacement equations of motion in the linear theory of homogeneous elastic solids, with reference to rectangular Cartesian coordinates and in the usual indicial notation**, are

$$c_{ijkl}u_{k,lj} + f_i = \rho u_i^{(2)}, \tag{1.1}$$

where

$$c_{ijkl} = c_{jikl} = c_{klij} . \tag{1.2}$$

The $u_i(\mathbf{x}, t)$ and $f_i(\mathbf{x}, t)$ are the Cartesian components of the displacement vector $\mathbf{u}(\mathbf{x}, t)$ and the body force $\mathbf{f}(\mathbf{x}, t)$; $\mathbf{x}$ is a point of the closed region of space R occupied by the body, and t represents the time. The c_{ijkl} are the components of the constant elasticity tensor $\mathbf{c}$ and ρ denotes the constant mass density.

This note is concerned exclusively with the displacement boundary-value problem of elastodynamics. We therefore adjoin the boundary condition

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^*(\mathbf{x}, t), \qquad (\mathbf{x}, t) \, \varepsilon \, B \times [0, \infty), \tag{1.3}$$

as well as the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}^0(\mathbf{x}), \qquad \mathbf{u}^{(1)}(\mathbf{x}, 0) = \mathbf{v}^0(\mathbf{x}), \qquad \mathbf{x} \, \varepsilon \, R. \tag{1.4}$$

Here $B \times [0, \infty)$ is the Cartesian product of the boundary B of R and the time interval $[0, \infty)$, and $\mathbf{u}^*$, $\mathbf{u}^0$, and $\mathbf{v}^0$ are, respectively, the prescribed surface displacement, initial displacement, and initial velocity. We assume that $\mathbf{u}^*$, $\mathbf{u}^0$, $\mathbf{v}^0$, $\rho > 0$, and c_{ijkl} satisfying (1.2) are given. The *displacement problem* under consideration then consists in finding a twice continuously differentiable vector field $\mathbf{u}$ on $R \times [0, \infty)$ which satsifies (1.1), (1.3), and (1.4).

The classical uniqueness theorem of Neumann [1] for the displacement problem rests on the assumption that the elasticity tensor is *positive semi-definite*, i.e., that

$$c_{ijkl}e_{ij}e_{kl} \geq 0 \tag{1.5}$$

for every tensor† e_{ij} . If the material is isotropic‡ (1.5) is equivalent to the inequalities

$$\mu \geq 0, \qquad -1 \leq \sigma < \tfrac{1}{2} \tag{1.6}$$

**Subscripts range over the integers $(1, 2, 3)$, summation over repeated subscripts is implied, and subscripts preceded by a comma indicate differentiation with respect to the corresponding space coordinate. Superscripts denote time differentiation. Letters in boldface designate tensors.

†In view of (1.2) we need not require $\mathbf{e}$ to be symmetric.

‡i.e.,

$$c_{ijkl} = \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + 2\sigma(1 - 2\sigma)^{-1}\delta_{ij}\delta_{kl}],$$

where δ_{ij} is the Kronecker delta.

provided μ and σ denote the shear modulus and Poisson's ratio of the material. Gurtin and Sternberg [2] have shown that for the isotropic case (1.6) may be relaxed without loss of uniqueness. Specifically they show that (1.6) may be replaced by the less stringent assumption

$$\mu \geq 0, \quad -\infty < \sigma < \tfrac{1}{2}, \quad 1 \leq \sigma < \infty. \tag{1.7}$$

Our objective here is to extend this result to the anisotropic case. In particular, we show that uniqueness of the displacement problem holds provided $\mathbf{c}$ is *semi-strongly-eliptic*, i.e.,

$$c_{ijkl}\alpha_i\alpha_k\beta_j\beta l \geq 0 \tag{1.8}$$

for every pair of vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$. Under the assumption of isotropy (1.8) and (1.7) are equivalent. Moreover, for the general anisotropic case (1.8) is weaker than (1.5) since the latter condition implies the former, but not vice-versa.

Toupin and Bernstein [3] have shown that semi-strong-elipticity admits a simple intrinsic interpretation: an elastic body propagates plane waves with real speeds if and only if (1.8) holds.

2. The uniqueness theorem. In what follows we assume that R is a *bounded regular region of space*, i.e., the boundary of R consists of a finite number of non-intersecting closed regular surfaces, the latter term being used in the sense of Kellogg [4].

UNIQUENESS THEOREM. *There is at most one solution to the displacement problem provided the elasticity tensor is semi-strongly elliptic.*

Proof. In view of the linearity of (1.1), (1.3), and (1.4) it is sufficient to show that $\mathbf{f} = \mathbf{u}^* = \mathbf{u}^0 = \mathbf{v}^0 = \mathbf{0}$ implies $\mathbf{u} = \mathbf{0}$. Assume that the former holds. Recall the power identity

$$\int_B c_{ijkl}u_{k,l}n_j u_i^{(1)}(\mathbf{x},\,t)\,dx + \int_R f_i u_i^{(1)}(\mathbf{x},\,t)\,dx = U^{(1)}(t) + K^{(1)}(t) \quad (0 \leq t < \infty), \tag{2.1}*$$

$$U(t) = \frac{1}{2}\int_R c_{ijkl}u_{i,j}u_{k,l}(x,\,t)\,dx, \tag{2.2}$$

$$K(t) = \frac{1}{2}\int_R \rho u_i^{(1)}u_i^{(1)}(x,\,t)\,dx, \tag{2.3}$$

which may be verified using (1.1), (1.2), and the divergence theorem. This identity, together with $\mathbf{f} = \mathbf{u}^* = \mathbf{0}$ and the boundary condition (2.3), implies that $U^{(1)}(t) = K^{(1)}(t) = 0 \; (0 \leq t < \infty)$. Next, since $\mathbf{u}^0 = \mathbf{v}^0 = \mathbf{0}$, the initial condition (2.4) yields $U(0) = K(0) = 0$. Hence

$$U(t) + K(t) = 0 \quad (0 \leq t < \infty). \tag{2.4}$$

Suppose we could show that

$$U(t) \geq 0 \quad (0 \leq t < \infty). \tag{2.5}$$

It would then follow from (2.4) that $K(t) = 0 \; (0 \leq t < \infty)$ which, because of (2.3), would imply that $\mathbf{u}^{(1)} = 0$ and hence that $\mathbf{u}(\mathbf{x},\,t) = \mathbf{u}(\mathbf{x},\,0) = \mathbf{0}$ for every $(\mathbf{x},\,t)\,\varepsilon\,R \times [0,\,\infty)$. The proof therefore reduces to showing that the inequality (2.5) holds.

We establish (2.5) using a method due entirely to van Hove [5]. Extend the defini-

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*We use dx to denote integration with respect to $\mathbf{x}$ in both surface and volume integrals.

tion of $\mathbf{u}$ to $E \times [0, \infty)$, where E is the entire three-dimensional Euclidean space, by setting

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad [E - R] \times [0, \infty) \tag{2.6}$$

Then, by virtue of the assumed smoothness of $\mathbf{u}$, it follows that, for each $t \geq 0$, $u_i(\mathbf{x}, t)$ and $u_{i,j}(\mathbf{x}, t)$ are both absolutely and square integrable over E with respect to $\mathbf{x}$. Hence u_k and $u_{k,l}$ possess, respectively, the three-dimensional Fourier transforms η_k and η_{kl} defined by

$$\eta_k(\mathbf{x}, t) = (1/2\pi)^{3/2} \int_E e^{i x_i v_i} u_k(\mathbf{y}, t) \, d\mathbf{y}, \tag{2.7}$$

$$\eta_{kl}(\mathbf{x}, t) = (1/2\pi)^{3/2} \int_E e^{i x_i v_i} u_{k,l}(\mathbf{y}, t) \, d\mathbf{y}. \tag{2.8}$$

Consequently, by (2.7), (2.8), the divergence theorem, and the fact that $\mathbf{u} = 0$ on $B \times [0, \infty)$, it follows that

$$\eta_{kl}(\mathbf{x}, t) = -i x_k \eta_l(\mathbf{x}, t). \tag{2.9}$$

Next, a fundamental theorem of Fourier analysis (cf., e.g., Goldberg [5], Theorem 13E) yields

$$\int_E u_{i,j}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) \, d\mathbf{x} = \int_E \eta_{ij}^c \eta_{kl}(\mathbf{x}, t) \, d\mathbf{x}, \tag{2.10}$$

where the superscript c denotes the complex conjugate. The functions η_k are, in general, complex valued. Their real and imaginary parts will be denoted by η_k' and η_k''

$$\eta_k = \eta_k' + i \eta_k''. \tag{2.11}$$

Then (2.6), (2.9), (2.10), and (2.11) imply

$$\int_R u_{i,j}(\mathbf{x}, t) u_{k,l}(\mathbf{x}, t) \, d\mathbf{x} = \int_E [x_j x_l \{ \eta_i'(\mathbf{x}, t) \eta_k'(\mathbf{x}, t) + \eta_i''(\mathbf{x}, t) \eta_k''(\mathbf{x}, t) \}] \, d\mathbf{x}. \tag{2.12}$$

Now multiply (2.12) by c_{ijkl} and use (1.8) to verify (2.5). This completes the proof.

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