A MATRIX EQUATION RELATED TO A NON-OSCILLATION CRITERION AND LIAPUNOV STABILITY

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1. Introduction. In this note it is shown that recent work of the author on Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems, [6; §7, in particular], implies a result on the solution of an algebraic matrix equation that is intimately related to the existence of a Liapunov function for linear differential systems with constant coefficients (see, for example, Hahn [2; §8], Bellman [1; Ch. 13], or LaSalle and Lefschetz [4; §17]).

Matrix notation is used throughout; in particular, matrices of one column are termed vectors. The symbol E_n is used for the $n \times n$ identity matrix, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. The notation $M \geq N$ or $N \leq N$, $\{M > N$ or $N < M\}$, is used to signify that M and N are Hermitian matrices of the same dimensions and $M - N$ is non-negative, {positive}, definite.

The basic result of this paper is as follows.

THEOREM A. *If A and B are constant $n \times n$ matrices with B Hermitian and $B \geq 0$, while the $n \times n^2$ matrix*

$$||B \; AB \; A^2B \cdots A^{n-1}B|| \tag{1.1}$$

has rank n, then there exist Hermitian matrices $W_\infty \leq 0$ and $W_{-\infty} \geq 0$ that are extreme solutions of the matrix equation

$$WA + A^*W + WBW = 0 \tag{1.2}$$

in the sense that $W = W_\infty$ and $W = W_{-\infty}$ are individually solutions of (1.2), *while if W is any Hermitian matrix satisfying* (1.2) *then $W_\infty \leq W \leq W_{-\infty}$. Moreover, $W_{-\infty} > 0$, $\{W_\infty < 0\}$, if and only if all proper values λ of A, $\{-A\}$, have $Re\, \lambda < 0$.*

The proof of Theorem A is presented in Sect. 2, and Sect. 3 contains a modification of this result that holds when (1.1) has rank $n - k$, $0 < k < n$. It is worth noting that the condition that a matrix (1.1) have rank n has appeared in certain treatments of optimal control problems. In particular, this condition is equivalent to the requirement that the autonomous control problem

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.3}$$

be *proper* in the sense of LaSalle [5; §4], or is *completely controllable* in the terminology of Kalman, Ho and Narendra in [3].

2. Proof of Theorem A. By Theorem 7.2 of Reid [6], a linear vector differential system

$$u' = Au + Bv, \qquad v' = Cu - A^*v, \tag{2.1}$$

with constant coefficient matrices satisfying

$$C^* = C, \qquad B^* = B \geq 0, \tag{2.2}$$

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and which is identically normal, is non-oscillatory on $(-\infty, \infty)$ if and only if there is an Hermitian matrix W satisfying the algebraic matrix equation

$$WA + A^*W + WBW - C = 0. \tag{2.3}$$

Moreover, if such a system (2.1) is non-oscillatory on $(-\infty, \infty)$ then there exist Hermitian matrices W_∞ and $W_{-\infty}$ which are individually solutions of (2.3), and are extreme solutions for the Riccati differential equation

$$W'(x) + W(x)A + A^*W(x) + W(x)BW(x) - C = 0 \tag{2.4}$$

in the sense that if $W(x)$ is any Hermitian solution of (2.4) on $(-\infty, \infty)$ then $W_\infty \leq W(x) \leq W_{-\infty}$; in particular, if W is any Hermitian solution of (2.3) then $W_\infty \leq W \leq W_{-\infty}$. Moreover, if $(U_0(x), V_0(x))$ is the solution of the corresponding matrix differential system

$$U' = AU + BV, \qquad V' = CU - A^*V, \tag{2.5}$$

satisfying $U_0(0) = 0$, $V_0(0) = E_n$, then $U_0(x)$ is non-singular for $x \neq 0$ and $W_0(x) = V_0(x)U_0^{-1}(x)$ is such that $W_0(x) \to W_\infty$, $\{W_0(x) \to W_{-\infty}\}$, as $x \to -\infty$, $\{x \to \infty\}$.

By definition, a system (2.1) is identically normal if the only solution $(u(x), v(x))$ of this system with $u(x) \equiv 0$ on a non-degenerate interval is the identically vanishing solution $u(x) \equiv 0$, $v(x) \equiv 0$. This condition is clearly equivalent to the condition that the fundamental matrix $Z(x) = \exp\{-xA^*\}$ of $z' = -A^*z$ is such that for any constant vector ξ the vector function $BZ(x)\xi$ is identically zero on a nondegenerate interval only if $\xi = 0$, and in view of the Cayley-Hamilton theorem one has the following result.

LEMMA 1. *A system* (2.1) *with constant coefficient matrices satisfying* (2.2) *is identically normal if and only if the matrix* (1.1) *is of rank n.*

Note that if η is a proper vector of A^* corresponding to a proper value λ, and $B\eta = 0$, then $BA^{*j}\eta = 0$, $(j = 1, 2, \cdots)$, so that the following result is immediate.

LEMMA 2. *If A and B are constant $n \times n$ matrices with $B \geq 0$, the matrix* (1.1) *is of rank n, and η is a proper vector of A^*, then $\eta^*B\eta > 0$.*

Now consider a system (2.1) for which condition (2.2) holds and also $C \geq 0$. Such a system is non-oscillatory on $(-\infty, \infty)$; that is, if $(u(x), v(x))$ is a solution with $u(x_1) = 0 = u(x_2)$, $(x_1 < x_2)$, then $u(x) \equiv 0$ on $[x_1, x_2]$. This result, which is a special case of Theorem 5.2 of Reid [6], is a direct consequence of the observation that if $(u(x), v(x))$ is such a solution then

$$0 = v^*(x)u(x) \Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} [v^*(x)Bv(x) + u^*(x)Cu(x)]\, dx;$$

hence $Cu(x) = 0$ and $Bv(x) = 0$ throughout $[x_1, x_2]$, so that $u' = Au$ and $u(x) \equiv 0$ as $u(x_1) = 0$.

In particular, consider a system

$$u' = Au + Bv, \qquad v' = -A^*v, \tag{2.1$_0$}$$

where A and B are $n \times n$ matrices with $B^* = B \geq 0$; that is, a system (2.1) satisfying (2.2) and with $C = 0$. In view of the above result and Lemma 1, it follows that under the hypotheses of Theorem A the system (2.1$_0$) is identically normal and non-oscillatory on $(-\infty, \infty)$, and thus by Theorem 7.2 of Reid [6] there exist Hermitian matrices W_∞, $W_{-\infty}$ which are individually solutions of (1.2), and such that if W is any Hermitian solution of this matrix equation then $W_\infty \le W \le W_{-\infty}$; in particular, $W = 0$ is an Hermitian solution of (1.2) so that $W_\infty \le 0 \le W_{-\infty}$. Finally, for $C = 0$ the solution $(U_0(x), V_0(x))$ of (2.5) satisfying $U_0(0) = 0$, $V_0(0) = E_n$ is

$$V_0(x) = e^{-xA^*}, \qquad U_0(x) = V_0^{*-1}(x) \int_0^x V_0^*(s) B V_0(s)\, ds, \tag{2.6}$$

so that for $x \neq 0$,

$$\begin{aligned} W_0(x) = V_0(x) U_0^{-1}(x) &= V_0(x) \left[\int_0^x V_0^*(s) B V_0(s)\, ds \right]^{-1} V_0^*(x), \\ &= \left[\int_0^x V_0^{*-1}(x) V_0^*(s) B V_0(s) V_0^{-1}(x)\, ds \right]^{-1}, \end{aligned}$$

and hence for $x \neq 0$,

$$W_0(x) = \left[\int_0^x e^{(x-s)A} B e^{(x-s)A^*}\, ds \right]^{-1} = \left[\int_0^x e^{tA} B e^{tA^*}\, dt \right]^{-1}.$$

In particular, $W_0(x_1) > W_0(x_2) > 0$ for $0 < x_1 < x_2$, and $W_{-\infty} = \lim_{x\to\infty} W_0(x)$ is positive definite if and only if

$$\lim_{x\to\infty} \eta^* \left[\int_0^x e^{tA} B e^{tA^*}\, dt \right] \eta$$

is finite for arbitrary constant vectors η. Now for η a proper vector of A^* corresponding to a proper value $\lambda = \alpha + i\beta$ we have

$$\eta^* \left[\int_0^x e^{tA} B e^{tA^*}\, dt \right] \eta = (\eta^* B \eta) \int_0^x e^{2\alpha t}\, dt,$$

and consequently if $W_{-\infty}$ is positive definite then all proper values λ of A^* have $Re\, \lambda < 0$; as λ_0 is a proper value of A^* if and only if $\bar{\lambda}_0$ is a proper value of A, this condition is equivalent to $Re\, \lambda < 0$ for all proper values λ of A. Conversely, if $Re\, \lambda < 0$ for all proper values λ of A, then

$$\int_0^\infty e^{tA} B e^{tA^*}\, dt = \lim_{x\to\infty} \int_0^x e^{tA} B e^{tA^*}\, dt$$

is a positive definite matrix, and $W_{-\infty}$, the inverse of this matrix, is positive definite. The conclusion that $W_\infty < 0$ if and only if all proper values λ of $-A$ have $Re\, \lambda < 0$ follows by a similar argument.

Clearly W is a positive definite solution of (1.2) if and only if $\Omega = W^{-1}$ is a positive definite solution of

$$A\Omega + \Omega A^* = -B. \tag{2.7}$$

For A and B real matrices, and B positive definite, (2.7) is a well-known matrix equation occurring in the determination of a Liapunov function for a linear vector differential equation with constant coefficients, (see, for example, Hahn [2; §8], Bellman [1; Ch. 13], or LaSalle and Lefschetz [4; §17]).

3. An extension of Theorem A. If the matrix (1.1) is not of rank n then the system (2.1) is not identically normal, and the result of Theorem A is modified. Now for non-oscillatory systems that are not identically normal the author [7] has recently discussed the existence of principal solutions and related distinguished solutions of associated Riccati equations, and the results of that paper might be used to obtain a generalization of Theorem A. For the case of the above systems with constant coefficients, however, this may be done directly as follows.

THEOREM B. *Suppose that* A *and* B *are constant* $n \times n$ *matrices with* B *Hermitian and* $B \geq 0$, *while the matrix* (1.1) *has rank* $n - k$, $0 < k < n$. *If* Δ *is an* $n \times k$ *matrix of rank* k *such that*

$$0 = \Delta^*B = \Delta^*A^jB, \qquad (j = 1, 2, \cdots, n - 1), \tag{3.1}$$

and Q *is an* $n \times (n - k)$ *matrix such that* $\Delta^*Q = 0$, $Q^*Q = E_{n-k}$, *and* $\mathcal{A} = Q^*AQ$, $\mathcal{B} = Q^*BQ$, *then there exist* $(n - k) \times (n - k)$ *Hermitian matrices* $\mathcal{W}_\infty \leq 0$, $\mathcal{W}_{-\infty} \geq 0$ *which are extreme solutions of the matrix equation*

$$\mathcal{W}\mathcal{A} + \mathcal{A}^*\mathcal{W} + \mathcal{W}\mathcal{B}\mathcal{W} = 0 \tag{3.2}$$

in the sense that $\mathcal{W} = \mathcal{W}_\infty$ *and* $\mathcal{W} = \mathcal{W}_{-\infty}$ *are individually solutions of* (3.2), *and if* $\mathcal{W}$ *is any Hermitian matrix satisfying* (3.2) *then* $\mathcal{W}_\infty \leq \mathcal{W} \leq \mathcal{W}_{-\infty}$; *moreover,* $\mathcal{W}_{-\infty} > 0$, $\{\mathcal{W}_\infty < 0\}$, *if and only if all proper values* λ *of* $\mathcal{A}$, $\{-\mathcal{A}\}$, *have Re* $\lambda < 0$.

Without loss of generality it may be supposed that the matrix Δ satisfying (3.1) is normalized so that $\Delta^*\Delta = E_k$; in this case, $E_n = QQ^* + \Delta\Delta^*$, so that $\mathcal{B}\mathcal{A}^* = Q^*B[QQ^*]A^*Q = Q^*B[QQ^* + \Delta\Delta^*]A^*Q = Q^*BA^*Q$, and by induction it follows that $\mathcal{B}\mathcal{A}^{*j} = Q^*BA^{*j}Q$, $(j = 1, 2, \cdots)$. If η is a vector such that $0 = \mathcal{B}\eta = \mathcal{B}\mathcal{A}^{*j}\eta$, $(j = 1, 2, \cdots)$, then $\xi = Q\eta$ is such that $0 = Q^*B\xi = Q^*BA^{*j}\xi$, $(j = 1, 2, \cdots)$, and as $\Delta^*B = 0$ it follows that $0 = B\xi = BA^{*j}\xi$, $(j = 1, 2, \cdots)$. Since $\Delta^*\xi = \Delta^*Q\eta = 0$, it then follows that $\xi = 0$, and hence $\eta = Q^*\xi = 0$. That is, the set of column vectors of $\mathcal{B}$, $\mathcal{B}\mathcal{A}^{*j}$, $(j = 1, 2, \cdots)$, contains $n - k$ linearly independent vectors, so that by the Cayley-Hamilton theorem the $(n - k) \times (n - k)^2$ matrix

$$\|\mathcal{B}\ \mathcal{A}\mathcal{B}\ \mathcal{A}^2\mathcal{B} \cdots \mathcal{A}^{n-k-1}\mathcal{B}\| \tag{3.3}$$

has rank $n - k$, and consequently the result of Theorem B is an immediate corollary to Theorem A.

The basic properties that relate the conclusion of Theorem B to the general results of Reid [7] are as follows:

(i) if $(U_0(x), V_0(x))$ is the solution of (2.1_0) defined by (2.6), then under the hypotheses of Theorem B the $n \times n$ matrix

$$K(x) = \int_0^x V_0^*(s - x_0)BV_0(s - x_0)\, ds, \qquad x \neq 0,$$

is an Hermitian matrix of rank $n - k$ and $K(x)\Delta = 0$, where Δ if an $n \times k$ matrix of rank k satisfying (3.1);

(ii) if K is an Hermitian $n \times n$ matrix of rank $n - k$, $0 < k < n$, Δ is an $n \times k$ matrix such that $K\Delta = 0$, $\Delta^*\Delta = E_k$, and Q is an $n \times (n - k)$ matrix such that $\Delta^*Q = 0$, $Q^*Q = E_{n-k}$, then the E. H. Moore general reciprocal $K^\#$ of K, (see, for example, Reid [7; §6]), is given by $K^\# = Q[Q^*KQ]^{-1}Q^*$.

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CONDITIONS FOR THE CAUSALITY OF NONLINEAR OPERATORS DEFINED ON A FUNCTION SPACE*

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Abstract. This note considers nonlinear operators T with both range and domain subsets of the space of complex n-vector-valued functions of the real variable t for $-\infty < t < \infty$. Conditions, in which energy-type quantities play a key role, are presented under which T is causal in the following sense:

Let $\mathfrak{D}(T)$ denote the domain of T and let $t_0 = \sup \{t' \mid \text{for all } f \in \mathfrak{D}(T), f(t) = 0 \text{ for almost all } t < t'\}$. Then T is causal if for an arbitrary $\delta > t_0$, $Tf = Tg$ a.e. on (t_0, δ) whenever f and g belong to $\mathfrak{D}(T)$ and $f = g$ a.e. on (t_0, δ).

1. Notation. Let $\mathcal{K}_n$ denote the space of complex measurable n-vector-valued functions of the real variable t for $-\infty < t < \infty$. The complex-conjugate transpose of an arbitrary $f \in \mathcal{K}_n$ is written as f^*. With g and h arbitrary elements of $\mathcal{K}_n$, and x either an arbitrary real number or ∞, let $\langle g, h; x\rangle$ denote

$$\int_{-\infty}^{x} g^* h \, dt,$$

and let

$$||g; x|| = (\langle g, g; x\rangle)^{1/2}.$$

The symbol $\mathfrak{R}$ denotes the set of real-valued functions and

$$\mathfrak{L}_{2n} = \{f \mid f \in \mathcal{K}_n, \langle f, f; \infty\rangle < \infty\}.$$

If $f \in \mathcal{K}_n$, and $x < \infty$, then f_x is defined by

$$f_x = f, \qquad t \leq x$$
$$f_x = 0, \qquad t > x.$$

2. Introduction. The external properties of a physical system can frequently be characterized by an operator relation of the form

$$g = Tf,$$

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