SOLUTION TO A CLASS OF COMPLETE HEAT FLOW EQUATIONS*

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Standard treatments of the heat flow equation:

$$\frac{\partial}{\partial t} u(\mathbf{R}, t) = \frac{\kappa}{\rho s} \nabla^2 u(\mathbf{R}, t) + \frac{A(\mathbf{R}, t)}{\rho s} \tag{1}$$

are confined to discussions of the two particular forms represented by Poisson's equation and the diffusion equation (see, for instance, [1], [2]). The purpose of this paper is to present a solution to a class of equations (1) in complete form. The solution appears distinctive from any yet set forth and serves to describe systems that solutions to the incomplete forms of (1) would not represent accurately.

Assume that the following conditions maintain in an application of equation (1):
(a) κ, ρ, and s are constants of the system;
(b) the boundaries of the system are maintained at the initial equilibrium temperature T_0;
(c) the heat input function is of the form $-A(\mathbf{R}, t) = \rho s F(\mathbf{R}) G(t)$, where $F(\mathbf{R})$ and $G(t)$ are integrable functions of position and time respectively.

We claim that the solution to equation (1) under these conditions may be expressed as:

$$u(\mathbf{R}, t) = T_0 + \sum_{l,m,n} K_{l,m,n} \left[\int_0^t G(\tau) e^{\alpha_{l,m,n}\tau} \, d\tau \right] e^{-\alpha_{l,m,n} t} E_{l,m,n}(\mathbf{R}), \tag{2}$$

where

$$\nabla^2 E_{l,m,n}(\mathbf{R}) + \frac{\rho s}{\kappa} \alpha_{l,m,n} E_{l,m,n}(\mathbf{R}) = 0, \tag{2'}$$

$$K_{l,m,n} = \int F(\mathbf{R}) E_{l,m,n}(\mathbf{R}) \, dV \Big/ \int E_{l,m,n}(\mathbf{R}) \, dV. \tag{2''}$$

Equation (2′) is solved in the usual manner ([1], p. 230) determining the $E_{l,m,n}(\mathbf{R})$ and $\alpha_{l,m,n}$ in accord with the (zero) boundary conditions on $F(\mathbf{R})$. The integrals in (2″) are carried out over the volume of the system. In virtue of the properties of the solutions to Eq. (2′) ([2], p.178), (2″) implies the equivalent equation

$$F(\mathbf{R}) = \sum_{l,m,n} K_{l,m,n} E_{l,m,n}(\mathbf{R}). \tag{2''}$$

To verify this solution it is only necessary to substitute (2) into (1), taking account of (2′) and (2″). Since Eq. (1) and all initial conditions and boundary conditions are satisfied, Eqs. (2), (2′), and (2″) constitute a solution to the specified class of equations (1).

1. H. Margenau and G. M. Murphy, *The Mathematics of Physics and Chemistry*, 2nd edition, D. Van Nostrand Co., N. Y., 1956
2. A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, N. Y., 1949

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