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THE USE OF THE PERTURBATION METHOD FOR DETERMINING THE PUMPING FREQUENCIES OF PARAMETRIC AMPLIFIERS*

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Abstract. This paper demonstrates that an old but very powerful and very general mathematical technique, the perturbation method, may be used to investigate such a modern application as parametric amplification. All possible pumping frequencies are predicted without recourse to any specialized procedures.

Introduction. One of the most powerful mathematical techniques for solving nonlinear differential equations, or linear differential equations with time varying coefficients is the perturbation method as introduced by Poincaré [1] and modified by Lindstedt [2]. In spite of this well known fact it is not used as often as it might be because often some less general but less lengthy means of solution [3] is available. If only the first order approximation is required, the method of Kryloff and Bogoliuboff may be used. If higher order approximations are required the KB method as generalized by Bogoliuboff and Mitropolsky [5] is useful. An excellent discussion of all methods can be found in the recent book of Minorsky [6]. It is often more difficult for the uninitiated in a particular field to learn and apply a specialized technique then it is for him to utilize an old familiar procedure. In the hope of making the perturbation method an even more familiar procedure, we apply it to the problem of parametric amplification.

The parametric amplifier makes use of a nonlinear reactive element to achieve amplification; the name arises, because during operation, a parameter of the circuit is made to vary with time. The recent interest in this type of amplifier is due to its noise-free operation and its ability to function at the higher radio frequencies. Thus, today the reactive element is often a nonlinear (per se) capacitor. In the earlier electrical experiments, at low frequencies, the varying parameter was either a capacitor or an inductor whose value was changed by mechanical means.

The degenerate case. As a review of the perturbation method, consider the so called degenerate case represented by a single circuit as shown in Fig. 1, in which R and C are constants but E and L are periodic functions of time. By Kirchhoff's second law we have

$$\frac{d}{dt}(Li) + Ri + \frac{1}{C}\int i\,dt = E. \tag{1.1}$$

In Eq. (1) let

$$L = L_0(1 + K\sin 2\omega t), \qquad L_0 = \text{const.}, \qquad 0 < K < 1,$$
$$R = Kr,$$
$$E = KA\sin\omega t + KB\cos\omega t.$$

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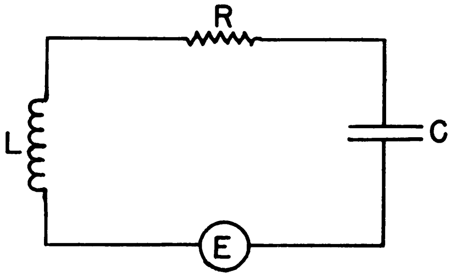

FIG. 1. The single circuit of the degenerate case.

Upon differentiating (1.1) we have the defining differential equation for the system, namely

$$L_0(1 + K \sin 2\omega t) \frac{d^2 i}{dt^2} + 4K\omega L_0 \cos 2\omega t \frac{di}{dt} + Kr \frac{di}{dt} + \left(\frac{1}{C} - 4\omega^2 L_0 K \sin 2\omega t\right) i$$
$$= KA\omega \cos \omega t - KB\omega \sin \omega t. \qquad (1.2)$$

From (1.1) we note that the input signal is represented by

$$E = KA \sin \omega t + KB \cos \omega t = KI \sin (\omega t + \varphi). \qquad (1.3)$$

Thus the input is periodic and has an angular frequency ω. Our problem is to find a periodic solution to (1.2) having this same frequency. We propose to do this by means of the perturbation method. For this purpose we assume that K is the perturbation parameter and write:

$$i = i_0 + Ki_1 + K^2 i_2 + \cdots, \qquad \omega = \omega_0 + K\omega_1 + K^2\omega_2 + \cdots. \qquad (1.4)$$

For convenience, in (1.2), let $\omega t = \Phi$. Equation (1.2) becomes

$$\omega^2 L_0(1 + K \sin 2\Phi) \frac{d^2 i}{d\Phi^2} + 4\omega^2 K L_0 \cos 2\Phi \frac{di}{d\Phi} + \omega Kr \frac{di}{d\Phi} + \left(\frac{1}{C} - 4\omega^2 L_0 K \sin 2\Phi\right) i$$
$$= KA\omega \cos \Phi - KB\omega \sin \Phi. \qquad (1.5)$$

We now substitute (1.4) into (1.5) with the result

$$\{\omega_0^2 + 2K\omega_0\omega_1 + K^2(\omega_1^2 + 2\omega_0\omega_2 + \cdots)\} L_0(1 + K \sin 2\Phi)(i_0'' + Ki_1''$$
$$+ K^2 i_2'' + \cdots) + 4\{\omega_0^2 + 2K\omega_0\omega_1 + K^2(\omega_1^2 + 2\omega_0\omega_2)\},$$
$$KL_0 \cos 2\Phi(i_0' + Ki_1' + K^2 i_2' + \cdots)$$
$$+ Kr(\omega_0 + K\omega_1 + K^2\omega_2 \cdots)(i_0' + Ki_1' + K^2 i_2' + \cdots)$$
$$+ \left\{\frac{1}{C} - 4[\omega_0^2 + 2K\omega_0\omega_1 + K^2(\omega_1^2 + 2\omega_0\omega_2) + \cdots]\right.$$
$$\left. \cdot L_0 K \sin 2\Phi\right\}(i_0 + Ki_1 + K^2 i_2 + \cdots)$$
$$= KA(\omega_0 + K\omega_1 + K^2\omega^2 \cdots) \cos \Phi - KB(\omega_0 + K\omega_1 + K^2\omega_2 + \cdots) \sin \Phi \qquad (1.6)$$

In (1.6), $i' = di/d\Phi$, $i'' = d^2 i/d\Phi^2$.

Let us now gather the coefficients of each power of K and set them individually equal to zero. The coefficient of K^0 gives us the equation

$$\omega_0^2 L_0 \frac{d^2 i_0}{d\Phi^2} + \frac{1}{C} i_0 = 0. \tag{1.7}$$

We note that since $\Phi = \omega t$, we want a periodic solution to Equation (1.7) with a period of 2π. Thus our solution must be of the form

$$i_0 = G \sin \Phi + H \cos \Phi, \tag{1.8}$$

and (1.7) will have this periodic solution provided

$$\omega_0^2 = 1/CL_0 . \tag{1.9}$$

This condition is imposed because we requested a solution having the same frequency as the input signal.

The coefficient of K^1 gives us the equation

$$i_1'' + \frac{1}{\omega_0^2 L_0 C} i_1 = -\left(\sin 2\Phi + \frac{2\omega_1}{\omega_0}\right) i_0'' - \left(4 \cos 2\Phi + \frac{r}{\omega_0 L_0}\right) i_0' + (4 \sin 2\Phi) i_0 + \frac{A}{\omega_0 L_0} \cos \Phi - \frac{B}{\omega_0 L_0} \sin \Phi. \tag{1.10}$$

We must substitute into (1.10), i_0 and its derivatives as given by (1.8). This results in the equation

$$\begin{aligned} \frac{d^2 i_1}{d\Phi^2} + i_1 = {} & -\left(\sin 2\Phi + 2\frac{\omega_1}{\omega_0}\right)(-G \sin \Phi - H \cos \Phi) \\ & - \left(4 \cos 2\Phi + \frac{r}{\omega_0 L_0}\right)(G \cos \Phi - H \sin \Phi) \\ & + \frac{A}{\omega_0 L_0} \cos \Phi - \frac{B}{\omega_0 L_0} \sin \Phi + 4 \sin 2\Phi (G \sin \Phi + H \cos \Phi). \end{aligned} \tag{1.11}$$

Upon carrying out the multiplications in (1.11) we find

$$\begin{aligned} \frac{d^2 i_1}{d\Phi^2} + i_1 = {} & \left(\frac{2G\omega_1}{\omega_0} + \frac{rH}{\omega_0 L_0} - \frac{B}{\omega_0 L_0}\right) \sin \Phi \left(\frac{2H\omega_1}{\omega_0} - \frac{rG}{\omega_0 L_0} + \frac{A}{\omega_0 L_0}\right) \cos \Phi \\ & + 5G \sin 2\Phi \sin \Phi - 5H \sin 2\Phi \cos \Phi - 4H \cos 2\Phi \cos \Phi + 4H \cos 2\Phi \sin \Phi. \end{aligned} \tag{1.12}$$

Before we can integrate (1.12) we must replace the trigonometric product terms by trigonometric sums. Thus,

$$\begin{aligned} \sin 2\Phi \sin \Phi &= 2 \sin^2 \Phi \cos \Phi = 2(\cos \Phi - \cos^3 \Phi) \\ &= 2 \cos \Phi - 2(3/4 \cos \Phi + 1/4 \cos 3\Phi) \\ &= 1/2 \cos \Phi - 1/2 \cos 3\Phi. \end{aligned} \tag{1.13}$$

Likewise,

$$\begin{aligned} \sin 2\Phi \cos \Phi &= 1/2 \sin \Phi + 1/2 \sin 3\Phi, \\ \cos 2\Phi \cos \Phi &= +1/2 \cos \Phi + 1/2 \cos 3\Phi, \\ \cos 2\Phi \sin \Phi &= -1/2 \sin \Phi + 1/2 \sin 3\Phi. \end{aligned}$$

With these substitutions, (1.12) becomes

$$\frac{d^2 i_1}{d\Phi^2} + i_1 = \left(\frac{2G\omega_1}{\omega_0} + \frac{rH}{\omega_0 L_0} - \frac{B}{\omega_0 L_0} + \frac{H}{2}\right) \sin \Phi + \left(\frac{2H\omega_1}{\omega_0} - \frac{rG}{\omega_0 L_0} + \frac{A}{\omega_0 L_0} + \frac{G}{2}\right) \cos \Phi + \frac{9H}{2} \sin 3\Phi - \frac{9G}{2} \cos 3\Phi \tag{1.14}$$

The terms in sin Φ and cos Φ on the right side of (1.14) will ruin the periodicity of i_1. Therefore, we must require these secular terms to have zero coefficients. This results in the conditions

$$\begin{aligned} \frac{2G\omega_1}{\omega_0} + \frac{rH}{\omega_0 L_0} - \frac{B}{\omega_0 L_0} + \frac{H}{2} &= 0, \\ \frac{2H\omega_1}{\omega_0} - \frac{rG}{\omega_0 L_0} + \frac{A}{\omega_0 L_0} + \frac{G}{2} &= 0. \end{aligned} \tag{1.15}$$

If Eqs. (1.15) are satisfied, (1.14) becomes

$$\frac{d^2 i_1}{d\Phi^2} + i_1 = \frac{9H}{2} \sin 3\Phi - \frac{9G}{2} \cos 3\Phi, \tag{1.16}$$

and the periodic solution is

$$i_1 = -\frac{9}{16} H \sin 3\Phi + \frac{9}{16} G \cos 3\Phi. \tag{1.17}$$

Thus, to the first order in K our solution is

$$\begin{aligned} i &= G \sin \Phi + H \cos \Phi - \frac{9K}{16} H \sin 3\Phi + \frac{9K}{16} G \cos 3\Phi \\ &= G \sin \omega t + H \cos \omega t - \frac{9K}{16} H \sin 3\omega t + \frac{9K}{16} G \cos 3\omega t, \end{aligned} \tag{1.18}$$

where G and H are related to the coefficients of the system by (1.9) and (1.15). To obtain a unique solution when Eqs. (1.15) are solved for G and H, the determinant of the coefficients must be different from zero:

$$4\left(\frac{\omega_1}{\omega_0}\right)^2 + \frac{r^2}{\omega_0^2 L_0^2} - \frac{1}{4} \neq 0. \tag{1.19}$$

To realize some of the significance of this result consider the following special cases.

Case I.

1. The system operates at a constant frequency $\omega = \omega_0$, so that $\omega_1 = 0$.
2. The input is $KA \sin \omega t$, i.e. $B = 0$.

Equations (1.15) become:

$$\left(\frac{r}{\omega_0 L_0} - 1/2\right) G = \frac{A}{\omega_0 L_0}, \qquad H = 0. \tag{1.20}$$

and our solution has the form

$$i = \frac{A}{\left(r - \frac{\omega_0 L_0}{2}\right)} \sin \omega t + \frac{9K}{16} \frac{A}{\left(r - \frac{\omega_0 L_0}{2}\right)} \cos 3\omega t. \tag{1.21}$$

Since $K < 1$, the coefficient of the third harmonic is less than $\frac{9}{16}$ of the fundamental. We might thus write

$$i \cong \frac{A}{r - (\omega_0 L_0/2)} \sin \omega t. \tag{1.22}$$

Equation (1.22) represents the current output. The voltage output is given by

$$Ri = Kri = \frac{KrA}{r - \frac{\omega_0 L_0}{2}} \sin \omega t, \tag{1.23}$$

but the voltage input is given by

$$E = KA \sin \omega t. \tag{1.24}$$

Thus, the voltage gain is

$$\text{Gain} = \frac{KrA}{r - (\omega_0 L_0/2)} \times \frac{1}{KA} = \frac{1}{1 - (\omega_0 L_0/2r)} = \frac{1}{1 - (K\omega_0 L_0/2R)} \tag{1.25}$$

Letting $Q = \omega_0 L_0/R$ Equation (1.25) becomes:

$$\text{Gain} = \frac{1}{1 - (KQ/2)} \tag{1.26}$$

Thus, for Case I, the system is an amplifier provided $KQ/2 < 1$.

Case II.

1. The input is $KA \sin \omega t$, i.e. $B = 0$.
2. The system's frequency may vary from ω_0 by an amount of $\Delta\omega$.

We note from Eq. (1.4) that to the first power in K

$$\omega = \omega_0 + K\omega_1 = \omega_0 + \Delta\omega,$$

so that

$$\frac{\Delta\omega}{\omega_0} = K\left(\frac{\omega_1}{\omega_0}\right). \tag{1.27}$$

Letting $\alpha = \omega_1/\omega_0$ and $\beta = KQ/2$ we can rewrite Equations (1.15) as

$$\begin{aligned} 2\alpha G + \frac{rH}{\omega_0 L_0} - \frac{B}{\omega_0 L_0} + \frac{H}{2} &= 0, \\ 2\alpha H - \frac{rG}{\omega_0 L_0} + \frac{A}{\omega_0 L_0} + \frac{G}{2} &= 0. \end{aligned} \tag{1.28}$$

Since $\omega_0 L_0/R = \omega_0 L_0/Kr = Q$ and $B = 0$, Eqs. (1.28) become

$$2\alpha G + \frac{H}{KQ} + \frac{H}{2} = 0 \quad \text{or} \quad 2\alpha G + \frac{H}{2\beta} + \frac{H}{2} = 0,$$

$$2\alpha H - \frac{G}{KQ} + \frac{A}{RQ} + \frac{G}{2} = 0 \quad \text{or} \quad 2\alpha H - \frac{G}{2\beta} + \frac{(A/R)K}{2\beta} + \frac{G}{2} = 0. \tag{1.29}$$

Solving the first of Equations (1.29) for H, we find

$$H = \frac{-4\alpha G}{1 + (1/\beta)} = \frac{-4\alpha\beta G}{1 + \beta}. \tag{1.30}$$

Substituting this value of H into the second Eq. (1.29), we have

$$\left[\frac{-8\alpha^2\beta}{1 + \beta} - \frac{1}{2\beta} + \frac{1}{2}\right]G + \frac{K(A/R)}{2\beta} = 0. \tag{1.31}$$

Solution of this equation for G, yields

$$G = \frac{K(A/R)(1 + \beta)}{1 - \beta^2 + 16\alpha^2\beta^2}. \tag{1.32}$$

Substituting this value of G into Equation (1.30), we obtain

$$H = \frac{-4\alpha\beta K(AR)}{1 - \beta^2 + 16\alpha^2\beta^2}. \tag{1.33}$$

Thus,

$$(G^2 + H^2)^{1/2} = \frac{K(A/R)[(1 + \beta)^2 + 16\alpha^2\beta^2]^{1/2}}{1 - \beta^2 + 16\alpha^2\beta^2}, \tag{1.34}$$

and the gain is given by the expression

$$\text{Gain} = \frac{[(1 + \beta)^2 + 16^2\beta^2]^{1/2}}{1 - \beta^2 + 16\alpha^2\beta^2}. \tag{1.35}$$

The non-degenerate case. Pumping frequencies. A common form of the parametric amplifier is shown in Fig. 2. Its operation depends on the nonlinearity of the element

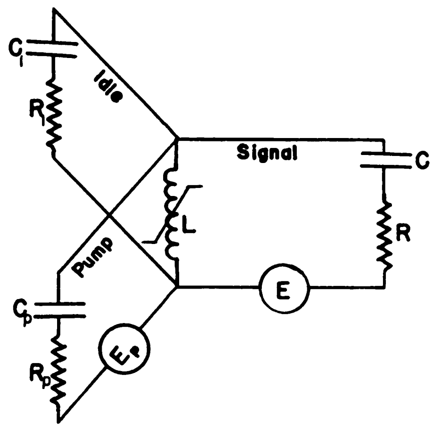

Fig. 2. The three circuits of the non-degenerate case.

common to all three circuits. In this case it is a nonlinear inductance. In other cases, it might be a nonlinear capacitance. Having chosen a nonlinear inductance, we realize that the nonlinearity is introduced by the B–H curve and that the shape of this curve determines, therefore, the characteristics of the amplifier. If we choose an analytical representation for the B–H curve, the characteristics of the amplifier will be determined by the parameters in this representation. Because of hysteresis, the B–H curve is actually a loop as shown in Fig. 3. If, however, the two branches of this loop are fairly close

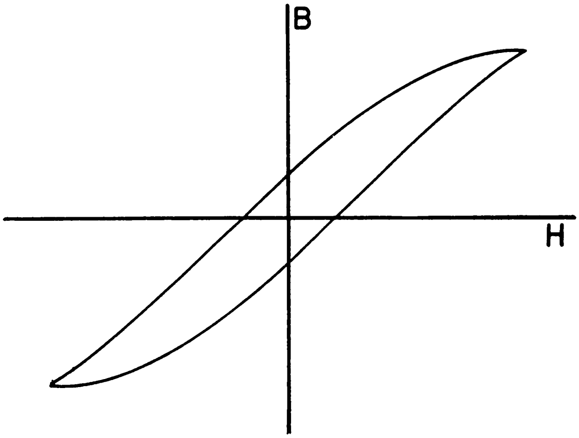

FIG. 3. The actual hysteresis loop of the system.

together, i.e., the hysteresis is small, we might proceed by considering a single average curve as shown in Fig. 4. This curve is an odd function and may be represented analytically as

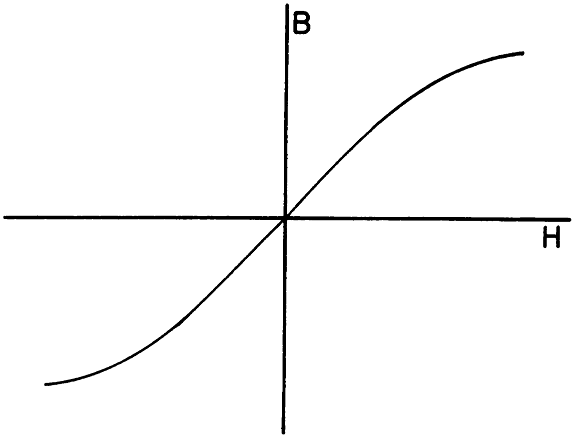

FIG. 4. The idealized B–H curve used to introduce nonlinearity into the system.

$$B = \mu_1 H + \mu_2 |H| H + \mu_3 H^3 + \mu_4 |H| H^3 + \cdots , \tag{2.1}$$

in which μ_1 , μ_2 $\cdots$ are constants. The characteristics of the amplifier should, therefore, be determined by these constants. In order to introduce the nonlinear inductance directly into the differential equations of the system, we remember that:

$$L = U_1 \frac{dB}{dH} \qquad L > 0 \tag{2.2}$$

Where U_1 is a constant whose value depends upon the units and configuration. Thus,

$$L = \mu_1 U_1 + 2\mu_2 U_1 |H| + 3\mu_3 U_1 H^2 + 4\mu_4 U_1 |H| H^2 + \cdots , \tag{2.3}$$

but

$$H = U_2 I, \tag{2.4}$$

where U_2 is a constant depending on units and configuration and I is the current through the nonlinear inductance. Substituting (2.4) into (2.3), we have

$$L = L_0 + L_1 |I| + L_2 I^2 + L_3 |I| I^2 + \cdots , \tag{2.5}$$

where $L_0 = \mu_1 U_1$, $L_1 = 2\mu_2 U_1 U_2$, $L_2 = 3\mu_3 U_1 U_2$, $L_3 = 4\mu_4 U_1 U_2$.

Thus,

$$L = L_0[1 + \Omega(I)],$$

where

$$\Omega(I) = \frac{L_1}{L_0} |I| + \frac{L_2}{L_0} I^2 + \frac{L_3}{L_0} |I| I^2 + \cdots , \tag{2.6}$$

The circuit of Fig. 2 consists of three loops and would, therefore, ordinarily be represented by three equations. However, in operation the branches are so tuned that each current remains essentially in its own branch and the sum of the three currents passes through the common nonlinear element. Thus, the signal loop can be described by the single equation

$$\frac{d}{dt}(LI) + Ri_s + \frac{1}{C}\int i_s \, dt = E \tag{2.7}$$

in which $L = L_0[1 + \Omega(I)]$ is the nonlinear inductance, I the total current through inductance $(i_s + i_i + i_p)$, i_s the current in signal loop, i_i the current in idle loop, i_p the current in pump loop, R the total resistance in signal loop, C the capacity in signal loop, E the applied voltage, i.e. signal input voltage. Equation (2.7) can be differentiated once more to give a differential equation of the form

$$L_0 \frac{d^2}{dt^2}[(i_s + i_i + i_p) + (i_s + i_i + i_p)\Omega(i_s + i_i + i_p)] + R\frac{di_s}{dt} + \frac{1}{C} i_s = \frac{dE}{dt}. \tag{2.8}$$

For this equation to describe the current i_s in the signal loop, the other currents, i_i and i_p must be predetermined. This can be achieved by making $i_p = E_p/Z_p$, where E_p is the voltage applied to the pump circuit and Z_p is the impedance of the pump circuit. If both of these quantities are known, i_p is known. Upon drawing the idle circuit as shown in Fig. 5, we note that the current i_i is determined by E_{NL}/Z_i' where E_{NL} is the potential across the nonlinear element and Z_i' is the impedance of the resistance and capacitance of the idle circuit. Z_i' is known and E_{NL} can be determined from

$$E_{NL} = \frac{d}{dt}(LI) = L_0 \frac{d}{dt}[I + I\Omega(I)]$$

$$= L_0 \frac{d}{dt}[(i_s + i_i + i_p) + (i_s + i_i + i_p)\Omega(i_s + i_i + i_p)]. \tag{2.9}$$

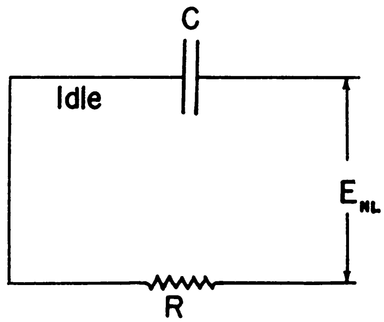

FIG. 5. The circuit used to determine the current in the idle loop.

Thus,

$$i_i = Z_i'^{-1} L_0 \frac{d}{dt} [(i_s + i_i + i_p) + (i_s + i_i + i_p)\Omega(i_s + i_i + i_p)] \tag{2.10}$$

If i_p is known, Eqs. (2.8) and (2.10) are a pair of simultaneous differential equations in i_s and i_i. Because of the tuning, we can write

$$\begin{aligned} i_s &= c_s \sin (\omega_s t + \Theta_s), \qquad i_i = c_i \sin (\omega_i t + \Theta_i), \\ i_p &= c_p \sin (\omega_p t + \Theta_p) = \frac{|E_p|}{Z_p} \sin (\omega_p t + \Theta_p), \end{aligned} \tag{2.11}$$

where $\omega_s \neq \omega_i \neq \omega_p$. Since i_i contains only the frequency ω_i, we are only interested in those terms on the right side of Eq. (2.10) which produce terms in ω_i. If now we assume a known relationship between the frequencies ω_s, ω_i, ω_p, we can then solve for i_i in terms of i_s which is unknown and i_p which is known. Thus Eq. (2.10) can be used to solve for i_i in terms of i_s. Upon substitution of this value of i_i, Eq. (2.8) becomes an equation in a single unknown, i_s. Writing Eq. (2.6) in the form

$$\begin{aligned} \Omega(i_s + i_i + i_p) = D_1 \, |i_s + i_i + i_p| + D_2(i_s + i_i + i_p)^2 \\ + D_3 \, |i_s + i_i + i_p| \, (i_s + i_i + i_p)^2 + \cdots , \end{aligned} \tag{2.12}$$

where

$$D_1 = \frac{L_1}{L_0} = \frac{2\mu_2 U_1 U_2}{\mu_1 U_1} = \frac{2\mu_2 U_2}{\mu_1},$$

$$D_2 = \frac{L_2}{L_0} = \frac{3\mu_3 U_1 U_2^2}{\mu_1 U_1} = \frac{3\mu_3 U_2^2}{\mu_1},$$

$$D_3 = \frac{L_3}{L_0} = \frac{4\mu_4 U_1 U_2^3}{\mu_1 U_1} = \frac{4\mu_4 U_2^3}{\mu_1},$$

We resort to the following obvious mode of attack. We examine Eq. (2.12) term by term, and for each term we determine whether or not there is a relationship between i_s, i_i, i_p such that the term in question will cause amplification when substituted in Eq. (2.8). Consider once again Eq. (2.7). After differentiation by t this becomes

$$\frac{d^2}{dt^2}(LI) + R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt}. \tag{2.13}$$

If we consider (2.12) term by term, the L used in (2.13) can be written as:

$$L_0 = (1 - K\,|I^n|), \tag{2.14}$$

where $n = 1, 2, \cdots$ and K is the parameter which introduces the nonlinearity. The minus sign is used because ordinarily L decreases as the current increases. Thus, (2.13) has the form

$$L_0 \frac{d^2}{dt^2}(I - KI\,|I^n|) + R\frac{di}{dt} + \frac{i}{C} = \frac{dE}{dt}. \tag{2.15}$$

We wish to examine (2.15) by means of the perturbation method using K as the perturbation parameter. To start from the free undamped oscillation we write

$$R = Kr, \qquad E = KE', \tag{2.16}$$

so that (2.15) becomes

$$L_0 \frac{d^2}{dt^2}(I - KI\,|I^n|) + Kr\frac{di}{dt} + \frac{i}{C} = K\frac{dE'}{dt}. \tag{2.17}$$

To save subscripts, $I = i_s + i_i + i_p$ is written as

$$I = i + f, \qquad \text{where} \quad i = i_s\,, \qquad f = i_i + i_p\,. \tag{2.18}$$

We now assume a solution of the form

$$i = i_0 + Ki_1 + K^2 i_2 + \cdots, \qquad \omega = \omega_0 + K\omega_1 + K^2\omega_2 + \cdots, \tag{2.19}$$

and an input of the form

$$E = KA\sin\omega t + KB\cos\omega t, \quad \text{i.e.} \quad E' = A\sin\omega t + B\cos\omega t. \tag{2.20}$$

With $\omega t = \Phi$, Eq. (2.17) becomes

$$\omega^2 L_0 \frac{d^2}{d\Phi^2}(I - KI\,|I^n|) + \omega Kr\frac{di}{d\Phi} + \frac{1}{C}i = \omega KA\cos\Phi - \omega KB\sin\Phi. \tag{2.21}$$

Into (2.21) we substitute (2.19) to obtain

$$\begin{aligned}
&[\omega_0^2 + 2K\omega_0\omega_1 + K^2(\omega_1^2 + 2\omega_0\omega_2 + \cdots)]\, L_0 \frac{d^2}{d\Phi^2}[f + i_0 + Ki_1 + K^2 i_2 + \cdots \\
&\quad - K(f + i_0 + Ki_1 + K^2 i_2 + \cdots)|\,(f + i_0 + Ki_1 + \cdots)|^n] \\
&\quad + \omega_0 + K\omega_1 + K^2\omega_2 + \cdots)\, Kr\frac{d}{d\Phi}(i_0 + Ki_1 + K^2 i_2 + \cdots) \\
&\quad + \frac{1}{C}(i_0 + Ki_1 + K^2 i_2 + \cdots) \\
&= (\omega_0 + K\omega_1 + K^2\omega_2 + \cdots)KA\cos\Phi - (\omega_0 + K\omega_1 + K^2\omega_2 + \cdots)KB\sin\Phi
\end{aligned} \tag{2.22}$$

Consider first the coefficient of K^0, which is

$$\omega_0^2 L_0 \frac{d^2}{d\Phi^2}(f + i_0) + \frac{1}{C}(i_0) = 0$$

or

$$\frac{d^2 i_0}{d\Phi^2} + \frac{1}{CL_0\omega_0^2}\, i_0 = -\frac{d^2 f}{d\Phi^2}. \tag{2.23}$$

Now

$$f = i_1 + i_p = i_i(\omega_i) + i_p(\omega_p) = f(\omega_i, \omega_p), \tag{2.24}$$

where ω_i is the frequency of current in the idle circuit, ω_p the frequency of current in the pump circuit, and $d^2f/d\phi^2 = \omega^{-2}d^2f/dt^2$ = function of ω_i and ω_p. The solution of (2.23) therefore contains the frequencies $\omega_0^{-1}(CL_0)^{-1/2}$, ω_i and ω_p. We remember, however, that the signal circuit is tuned so that it does not contain appreciable currents of frequencies ω_i and ω_p. Thus, by making ω_i and ω_p greatly different than $\omega_0^{-1}(CL_0)^{-1/2}$, Eq. (2.23) is essentially

$$\frac{d^2 i_0}{d\Phi^2} + \frac{1}{CL_0\omega_0^2}\, i_0 = 0, \tag{2.25}$$

In the variable Φ, the input has the frequency 1, and if we desire to have a solution with the same frequency we note from Eq. (2.25) that

$$\omega_0^2 = (CL_0)^{-1}. \tag{2.26}$$

Under these conditions, the solution of Eq. (2.25) is

$$i_0 = G \sin \Phi + H \cos \Phi, \tag{2.27}$$

and our assumed solution has the form

$$i = i_0 = G \sin \omega t + H \cos \omega t, \qquad \omega = \omega_0 = (CL_0)^{-1/2}. \tag{2.28}$$

The form of i_0 does not depend on K and, therefore, does not depend on Ω. The next step is to get a differential equation by setting the coefficient of K' to zero. This, of course, does depend on $\Omega(I)$. For the general term to the nth power having the coefficient K, the resulting differential equation is:

$$i_1'' + i_1 = \left(\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0} + \frac{2\omega_1 H}{\omega_0}\right)\cos \Phi + \left(\frac{-B}{\omega_0 L_0} + \frac{rH}{\omega_0 L_0} + \frac{2\omega_1 G}{\omega_0}\right)\sin \Phi$$
$$\frac{-2\omega_1}{\omega_0} f'' \qquad \frac{+d^2}{d\Phi^2}(f + i_0)^{n+1}, \tag{2.29}$$

where $i_1' = di_1/d\Phi$; $i_1'' = d^2i_1/d\Phi^2$, $f'' = d^2f/d\Phi^2$.

We must now insure that Eq. (2.29) has a periodic solution. This requires that the coefficients be zero for any terms on the right whose frequency is 1. The last two terms are

$$f = i_1 + i_p = c_i \sin(\omega_i t + \Theta_i) + c_p \sin(\omega_p t + \Theta_p),$$
$$f'' = \frac{1}{\omega^2}\frac{d^2 f}{dt^2} = \frac{-c_i\omega_i^2}{\omega^2}\sin(\omega_i t + \Theta_i)\ \frac{-c_p\omega_p^2}{\omega^2}\sin(\omega_p t + \Theta_p).$$

Thus, the term $-2\omega_1 f''/\omega_0$ will introduce currents of frequencies ω_i and ω_p. We have, of course, required that there be no such currents in the signal circuit. This can be realized in i_1 by making $\omega_1 = 0$. Thus Eq. (2.29) becomes

$$i_1'' + i_1 = \left(\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0}\right)\cos \Phi + \left(\frac{-B}{\omega_0 L_0} + \frac{rH}{\omega_0 L_0}\right)\sin \Phi + \frac{d^2}{d\Phi^2}(f + i_0)^{n+1}, \tag{2.30}$$

and interest centers on the last term.

If this term does not contain a current of frequency 1, our periodicity conditions become

$$A - rG = 0, \qquad -B + rH = 0, \tag{2.31}$$

but

$$\text{gain} = \frac{\text{output voltage}}{\text{input voltage}} = \frac{Kr(G \sin \omega t + H \cos \omega t) + Ki_1}{KA \sin \omega t + KB \cos \omega t}. \tag{2.32}$$

With the periodicity conditions (2.31) satisfied, and with $d^2(f + i_0)^{n+1}/d\phi^2$ containing no terms of frequency 1, Eq. (2.30) is essentially $i_1'' + i_1 = 0$ and (2.33) has a solution

$$i_1 = G_1 \sin \Phi + H_1 \cos \Phi, \tag{2.34}$$

in which G_1 and H_1 are to be determined by initial conditions. Suppose our initial conditions are

$$\text{at} \quad t = 0: \qquad i = H, \qquad i' = G.$$

These can be realized by making $i_0(0) = H, i_0'(0) = G, i_1(0) = i_2(0) = 0, i_1'(0) = i_2'(0) = 0$. Thus $G_1 = H_1 = 0$ and Eq. (2.34) becomes

$$i_1 = 0. \tag{2.35}$$

Equation (2.32) now assumes the form

$$\text{gain} = \frac{rG \sin \omega t + rH \cos \omega t}{A \sin \omega t + B \cos \omega t} = 1. \tag{2.36}$$

Therefore, to an approximation involving the first power of K, if the nonlinear expression

$$\frac{d^2}{d\Phi^2}(f + i_0)^{n+1}$$

does not contain a term of unit frequency, the system will not amplify. The next question is: how can we make this expression contain a term of unit frequency

Case I: $n = 1$.

$$(f + i_0)^{n+1} = (i_0 + f)^2 = [i_0 + (i_i + i_p)]^2 = i_0^2 + 2i_0 i_i + 2i_0 i_p + i_i^2 + i_p^2 + 2i_i i_p$$

$$i_0 = G \sin \omega t + H \cos \omega t = F \sin (\omega t + \Theta)$$

$$i_i = c_i \sin (\omega_i t + \Theta_i)$$

$$i_p = c_p \sin (\omega_p t + \Theta_p)$$

Thus the terms in the expansion will yield Fourier components of frequencies as follows:

$$i_0^2 \to 2\omega, \qquad i_0^2 \to 2\omega_i, \qquad i_p^2 \to 2\omega_p$$

$$2i_0 i_i \to \omega + \omega_i \text{ and } \omega - \omega_i, \quad 2i_0 i_p \to \omega + \omega_p \text{ and } \omega - \omega_p, \quad 2i_i i_p \to \omega_i + \omega_p \text{ and } \omega_i - \omega_p.$$

Case II: $n = 2$.

$$(f + i_0)^{n+1} = (i_0 + f)^3 = i_0^3 + 3i_0^2 f + 3i_0 f^2 + f^3$$
$$= i_0^3 + 3i_0^2(i_i + i_p) + 3i_0(i_i^2 + 2i_i i_p^2) + (i_i^3 + 3i_i^2 i_p + 3i_i i_p^2 + i_p^3).$$

$$i_0^3 \to \omega \text{ and } 3\omega, \quad 3i_0^2 i_i \to 2\omega + \omega_i \text{ and } 2\omega - \omega_i \text{ and } \omega_i,$$
$$3i_0^2 i_p \to 2\omega + \omega_p \text{ and } 2\omega - \omega_p \text{ and } \omega_p, \quad 3i_0 i_i^2 \to \omega + 2\omega_i \text{ and } \omega - 2\omega_i \text{ and } \omega,$$
$$6i_0 i_i i_p \to [\omega + (\omega_i + \omega_p)] \text{ and } [\omega - (\omega_i + \omega_p)],$$
$$3i_0 i_p^2 \to \omega + 2\omega_p \text{ and } \omega_p \text{ and } \omega - 2\omega_p \text{ and } \omega, \quad i_i^3 \to \omega_i \text{ and } 3\omega_i,$$
$$3i_i^2 i_p \to (2\omega_i + \omega_p) \text{ and } 2\omega_i - \omega_p \text{ and } \omega_p,$$
$$3i_i i_p^2 \to \omega_i + 2\omega_p \text{ and } \omega_i - 2\omega_p \text{ and } \omega_i, \quad i_p^3 \to \omega_p \text{ and } 3\omega_p.$$

Case III: $n = 3$.

$$(f + i_0)^{n+1} = (i_0 + f)^4 = i_0^4 + 4i_0^3 f + 6i_0^2 f^2 + 4i_0 f^3 + f^4$$
$$= i_0^4 + 4i_0^3(i_i + i_p) + 6i_0^2(i_i + i_p)^2 + 4i_0(i_i + i_p)^3 + (i_i + i_p)^4$$
$$(i_0 + f)^4 = i_0^4 + 4i_0^3 i_i + 4i_0^3 i_p + 6i_0^2(i_i^2 + 2i_i i_p + i_p)^2$$
$$+ 4i_0(i_i^3 + 3i_i^3 i_p + 3i_i i_p^2 + i_p^3) + (i_i + i_p)^4$$

Consider these terms one by one to determine the possible frequencies listed below.

$$i_0^4 \to (0 + 2\omega)(0 + 2\omega) = 2\omega, 4\omega, \quad 4i_0^3 i_i = (\omega + 3\omega)\omega_i \to \omega \pm \omega_i, 3\omega \pm \omega_i,$$
$$4i_0^3 i_p \to \omega \pm \omega_p, 3\omega \pm \omega_p, \quad 6i_0^2 i_i^2 = (0 + 2\omega)(0 + 2\omega_i) \to 0, 2\omega, 2\omega_i, 2\omega \pm \omega_i,$$
$$12i_0^2 i_i i_p = (0 + 2\omega)(\omega_i)(\omega_p) \to (\omega_i + 2\omega \pm \omega_i)(\omega_p) = \omega_i \pm \omega_p, (2\omega \pm \omega_i) \pm \omega_p$$
$$6i_0^2 i_p^2 \to 0, 2\omega, 2\omega_p, 2\omega \pm 2\omega_p, \quad 4i_0 i_i^3 \to \omega_i \pm \omega, 3\omega_i \pm \omega, \quad 12i_0 i_i i_p \to \omega \pm \omega_p, (2\omega_i \pm \omega) \pm \omega_p,$$
$$12i_0 i_i i_p^2 \to \omega_i \pm \omega, (2\omega_p \pm \omega_i) \pm \omega, \quad 4i_0 i_p^3 \to \omega_p \pm \omega, 3\omega_p \pm \omega,$$
$$i_i^4 \to 2\omega_i, 4\omega_i, \quad 4i_i^3 i_p \to \omega_i \pm \omega_p, 3\omega_i \pm \omega_p,$$
$$6i_i^2 i_p^2 \to 0, 2\omega_i, 2\omega_p, 2\omega_i \pm 2\omega_p, \quad 4i_i i_p^3 \to \omega_p \pm \omega_i, 3\omega_p \pm \omega_i, \quad i_p^4 \to 2\omega_p, 4\omega_p.$$

We see, therefore, that without further restrictions there are many ways to introduce terms of frequency ω. One way is to require that

$$\omega_p = \omega - \omega_i \tag{2.37}$$

without any fixed integral relationships between the frequencies. This results in a contribution from the term $2i_i i_p$ from Case I and the term $12i_0^2 i_i i_p$ from Case III. Assuming that the Case I term adequately represents the nonlinearity, we consider only the term

$$2i_i i_p = 2c_p c_i \sin(\omega_i t + \Theta_i) \sin(\omega_p t + \Theta_p)$$
$$= c_i c_p \{\cos[(\omega_i - \omega_p)t + (\Theta_i - \Theta_p)] - \cos[(\omega_i + \omega_p)t + (\Theta_i + \Theta_p)]\}.$$

Using condition (37), we have only to investigate the term

$$(f + i_0)^2 = -c_i c_p \cos(\omega t + \Theta_p + \Theta_i), \tag{2.38}$$

from which

$$\frac{d^2}{d\Phi^2}(f + i_0)^2 = +c_i c_p \cos(\Phi + \sigma)\sigma = \Theta_p + \Theta_i$$
$$= (+c_i c_p \cos\sigma - (c_i c_p \sin\sigma) \sin\Phi. \tag{2.39}$$

Thus, Eq. (2.30) becomes

$$i_1'' + i_1 = \left(\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0} + c_i c_p \cos\sigma\right)\cos\Phi + \left(\frac{-B}{\omega_0 L_0} + \frac{rH}{\omega_0 L_0} - c_i c_p \sin\sigma\right)\sin\Phi, \quad (2.40)$$

so that our periodicity conditions become

$$\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0} + c_i c_p \cos\sigma = 0, \qquad \frac{-B}{\omega_0 L_0} + \frac{rH}{\omega_0 L_0} - c_i c_p \sin\sigma = 0. \quad (2.41)$$

If we let 0 be the output voltage, then

$$|0| = Kr\sqrt{G^2 + H^2}$$

or

$$|0^2| = K^2 r^2 (G^2 + H^2). \quad (2.42)$$

From Eqs. (2.41), we deduce

$$rG = A - \kappa\cos\sigma, \qquad rH = B - \kappa\sin\sigma, \qquad \text{where} \quad \kappa = c_i c_p \omega_0 L_0. \quad (2.43)$$

Thus,

$$|0^2| = (A^2 - 2A\kappa\cos\sigma + \kappa^2\cos^2\sigma + B^2 - 2B\kappa\sin\sigma + \kappa^2\sin^2\sigma)K^2,$$

$$\frac{\partial}{\partial\sigma}|0^2| = (2A\kappa\sin\sigma - 2B\kappa\cos\sigma)K^2; \qquad \frac{\partial^2}{\partial\sigma^2}|0^2| = (2A\kappa\cos\sigma + 2B\kappa\sin\sigma)K^2, \quad (2.44)$$

and

$$\sin\sigma = -B(A^2 + B^2)^{-1/2}; \qquad \cos\sigma = -A(A^2 + B^2)^{-1/2} \quad (2.45)$$

is the best phase relationship in the sense that it will give the maximum output. We are given A and B by the input, and, therefore, Eq. (2.45) determines σ. The maximum amplitude of the current which flows in the pump circuit determines c_p. For any particular system c_p is determined by the voltage applied to the pump. Thus c_p is known. The value of c_i, the maximum amplitude of the current which flows in the idle circuit, is determined by the parameters of the idle circuit and the emf across the nonlinear element. The idle circuit is shown in Fig. 5.

The emf E_{NL} across the nonlinear element is given by

$$\frac{d}{dt}(LI) = \frac{d}{dt}[L_0(1 - KI)I] = L_0\frac{d}{dt}(I - KI^2) = E_{NL}, \quad (2.46)$$

where the current I through the nonlinear element is

$$I = i(\omega) + i_i(\omega_i) + i_p(\omega_p). \quad (2.47)$$

Thus,

$$E_{NL} = L_0\frac{d}{dt}[(i + i_i + i_p) - K(i + i_i + i_p)^2]. \quad (2.48)$$

The only part of Eq. (2.48) which will affect the idle circuit is the part which has a frequency ω_i. For our purposes, we therefore have

$$E_{NL} = L_0\frac{d}{dt}[i_i - K(i + i_i + i_p)^2]. \quad (2.49)$$

Consulting Case I and using Eq. (2.37), Eq. (2.49), we find

$$E_{NL} = L_0 \frac{d}{dt} [i_i - 2Kii_p]. \tag{2.50}$$

Substituting the values of the currents into (2.50) we have

$$E_{NL} = L_0 \frac{d}{dt} \{c_i \sin (\omega_i t + \Theta_i) - Kc_pF[\cos (\omega - \omega_p)t + \Theta - \Theta_p) - \cos (\omega + \omega_p)t + \Theta + \Theta_p]\}. \tag{2.51}$$

Setting $\omega - \omega_p = -\omega_i$, which means using Eq. (2.37), and discarding the last term, we may write (2.5) as

$$E_{NL} = L_0 \frac{d}{dt} \{c_i \sin (\omega_i t + \Theta_i) - Kc_pF \cos (-\omega_i t + \Theta - \Theta_p)\} = \omega_i c_i L_0 \cos \Phi_i + K\omega_i c_p F \sin (\Phi_i + \delta), \tag{2.52}$$

with $\Phi_i = \omega_i t + \Theta_i$ and $\delta = \Theta_p - \Theta - \Theta_i$ or

$$E_{NL} = [\omega_i c_i L_0 + K\omega_i c_p F \sin \delta] \cos \Phi_i + [K\omega_i c_p F \cos \delta] \sin \Phi_i. \tag{2.53}$$

To the first order in K, the magnitude of E_{NL} is given by

$$|E_{NL}| = [\omega_i^2 c_i^2 L_0^2 + 2\omega_i^2 c_i c_p KF \sin \delta + \cdots]^{1/2}. \tag{2.54}$$

Now

$$|E_{NL}| = \omega_i c_i \quad \text{or} \quad E_{NL}^2 = Z_i^2 c_i^2, \tag{2.55}$$

where Z_i is the impedance of the circuit shown in Fig. 5. From this figure, we see that

$$Z_i^2 = R_i^2 + \frac{1}{\omega_i^2 c_i^2} = R_i^2 + \omega_i^2 L_0^2 \quad \text{since usually} \quad \omega_i = (C_i L_0)^{-1/2}. \tag{2.56}$$

Combining Eqs. (2.54), (2.55), (2.56), we therefore have

$$\omega_i^2 c_i^2 L_0^2 + 2\omega_i^2 c_i c_p KF \sin \delta = R_i^2 c_i^2 + \omega_i^2 L_0^2 c_i^2$$

from which we find

$$c_i = (2K\omega_i^2 c_p F \sin \delta)/R_i^2 \tag{2.57}$$

if $c_i \neq 0$. Since c_p is known, we write

$$c_i c_p = KFM, \quad \text{where} \quad M = (2\omega_i^2 c_p^2 \sin \delta)/R_i^2. \tag{2.58}$$

Equations (2.41) can now be written as

$$\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0} + KFM \cos \sigma = 0, \qquad \frac{-B}{\omega_0 L_0} + \frac{rH}{\omega_0 L_0} - KFM \sin \sigma = 0. \tag{2.59}$$

Consider now the case of an input voltage

$$E = KA \sin \omega t; \qquad B = 0. \tag{2.60}$$

Equation (2.45) yields $\sigma = 0$, and the second Eq. (2.59) yields $H = 0$. We thus have left only the first Eq. (2.59) which becomes

$$\frac{A}{\omega_0 L_0} - \frac{rG}{\omega_0 L_0} + KGM = 0. \tag{2.61}$$

Equation (2.61) can be solved for G in the form

$$G = \frac{A}{r(1 - K^2QM)} \quad \text{where} \quad Q = \frac{\omega L_0}{R}. \tag{2.62}$$

From Eq. (2.42) the magnitude of the output voltage is KrG. Thus the gain is given by:

$$\text{gain} = \frac{1}{1 - K^2QM}. \tag{2.63}$$

Provided we supply a current through the pump and have nonlinearity in the inductance, we have an amplifier. We note the gain to this approximation does not depend on the input amplitude.

We have just completed an example of an amplifier pumped directly. They can, however, also be pumped subharmonically. If we return to the nonlinear terms introduced by Case III, we find listed the possible frequencies. Two terms interest us at this time:

$$6i_i^2 i_p^2 \to 2\omega_i \pm 2\omega_p \quad \text{and} \quad 4i_i i_p^3 \to 3\omega_p \pm \omega_i. \tag{2.64}$$

Suppose we choose pump and idle frequencies in such a manner that

$$2\omega_i \pm 2\omega_p = \pm\omega \quad \text{or specifically} \quad 2\omega_p = 2\omega_i + \omega. \tag{2.65}$$

If now we let $2\omega_p = \omega_p'$ and $2\omega_i = \omega_i'$, Eq. (2.65) becomes

$$\omega_p' = \omega_i' + \omega, \tag{2.66}$$

which is the subharmonic case presented in Fig. 1a of the report by Mortenson (7). If we use the second Eq. (2.64) and write it in the form

$$3\omega_p = \omega_i + \omega \quad \text{or} \quad \omega_p' = \omega_0 + \omega \quad \text{with} \quad \omega_p' = 3\omega_p, \tag{2.67}$$

we have the results given in Fig. 1b of this report. Thus, the perturbation technique leads once again to some well known results. By substituting the values of the currents into Equations (2.64) the gains could be solved for as in our previous example.

The perturbation method is so general and so powerful that it can be used to determine the pumping frequencies which result in amplification for all types of parametric amplifiers (8). It readily predicts the following well known cases:

$$\begin{aligned} \omega_p &= \omega - \omega_1 && \text{up converter type,} \\ \omega_p &= \omega + \omega_1 && \text{negative resistance type,} \\ \omega_p &= \omega + \omega && \text{degenerate type,} \\ 2\omega_p &= 2\omega_1 + \omega && \tfrac{1}{2} \text{ subharmonic pumping,} \\ 3\omega_p &= \omega_i + \omega && \tfrac{1}{3} \text{ subharmonic pumping.} \end{aligned}$$

Many other pumping frequencies are indicated and some might be worthy of further consideration. The important point here is that all are suggested by the same general method, namely the perturbation method.

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8. If the reader is interested in other cases, including stability and bandwidth, he may refer to—

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   For bandwidth see Part II, Page 10
   For stability see Part III, Page 21