ON THE SOLUTION OF A TRANSCENDENTAL EQUATION IN SCATTERING THEORY—PART II*

BY

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Introduction. The scattering theory of electromagnetic waves by an infinitely long dielectric cylinder of large radius has been considered by Franz and Beckmann [1], [2]. They found that the computation of the diffraction effects requires the solution of two characteristic equations in the complex ν-plane. The first equation,

$$x \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - y \frac{H_\nu^{(2)\prime}(y)}{H_\nu^{(2)}(y)} = 0,$$

was treated by Streifer and Kodis [3]; the second equation,

$$x \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - y \frac{J_\nu'(y)}{J_\nu(y)} = 0, \tag{1}$$

will be considered here. The parameters are $x = k_2 a$, $y = k_1 a$ where a is the cylinder radius, k_2 the outer wave number, and k_1 the wave number of the cylinder. As in Ref. [3], y is taken to be less than x so that the refractive index, $N = y/x$, is less than one.

Solution. The solutions of Eq. (1) will be developed as asymptotic series in x or y, when both are real and much greater than one, by means of appropriate approximations for the Bessel and Hankel functions. Since each approximation can be used only within certain ranges of the parameters ν, x, and y, the ν-plane must be divided into a number of regions, as shown in Fig. 1, and suitable expansions developed for each. The circular regions do not have specific boundaries since their radii are $|\nu \pm y| = O(y^{1/3})$ and $|\nu - x| = O(x^{1/3})$.

We consider only the situation in which the circular regions are well separated, i.e., $x - y = O(x)$. In that case solutions of Eq. (1) exist only near the solutions of $H_\nu^{(1)}(x) = 0$ and $J_\nu(y) = 0$, which lie on the solid curves of Fig. 1 and the real axis for $\nu < y$. Far from these curves and outside the circles both $H_\nu^{(1)}(x)$ and $J_\nu(y)$ may be accurately represented by a single exponential function (see the appendix), and the ratios in (1) are proportional to $(\nu^2 - x^2)^{1/2} + O(1)$ and $(\nu^2 - y^2)^{1/2} + O(1)$. After substitution in (1) it is found that

$$(\nu^2 - x^2)^{1/2} - (\nu^2 - y^2)^{1/2} + O(1) = 0,$$

which can be put into the form,

$$x^2 - y^2 = O(x).$$

Since $x - y = O(x)$, the existence of a root implies that $O(x^2) = O(x)$, which cannot be true for sufficiently large x.

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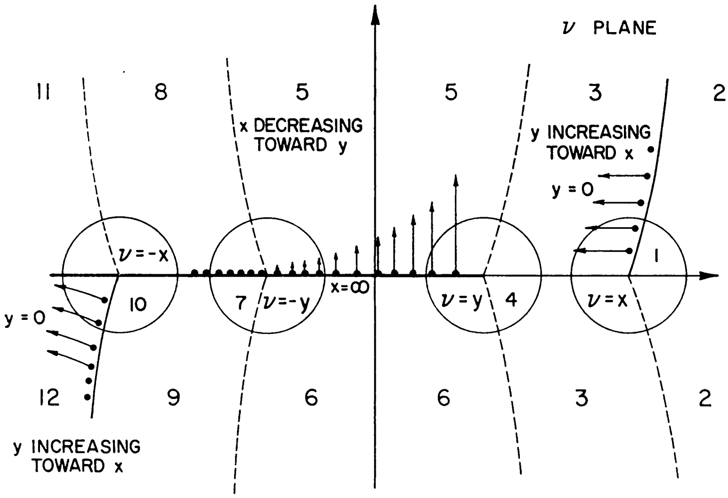

FIG. 1. Regions and roots in the ν-plane.

In region 1, where $|\nu - x| = O(x^{1/3})$ and $|\nu - y| = O(y)$, the appropriate asymptotic forms substituted into (1) yield

$$\frac{Ai'(-qe^{-i\pi/3})}{Ai\,(-qe^{-i\pi/3})} = \Gamma_1 \tag{2a}$$

and

$$\Gamma_1 = -\frac{e^{i\pi/3}}{\partial_x q}\left[\left(\frac{\nu^2}{x^2} - N^2\right)^{1/2} + \frac{yN}{2(\nu^2 - y^2)} + \frac{1}{3x} - \frac{\partial_x p}{p} + \frac{yN}{8(\nu^2 - y^2)^{3/2}}\left(1 - \frac{5\nu^2}{\nu^2 - y^2}\right) + O(x^{-3})\right], \tag{2b}$$

where p and q are defined in the Appendix, $Ai(\eta)$ is the Airy function of the first kind, and $Ai'(\eta) = dAi(\eta)/d\eta$.

The solution of (2a) is found by the methods of Ref. [3], and the resulting formula for ν is:

$$\nu = x - \eta_0 e^{i\pi/3}\left(\frac{x}{2}\right)^{1/3} + e^{i2\pi/3}L_{10}\left(\frac{2}{x}\right)^{1/3} + L_{30}\left(\frac{2}{x}\right) + e^{i\pi/3}L_{50}\left(\frac{2}{x}\right)^{5/3}$$
$$- \mu + e^{i\pi/3}L_{21}(\mu - \mu^3)\left(\frac{2}{x}\right)^{2/3} + e^{i2\pi/3}(L_{41}\mu + L_{43}\mu^3 + L_{45}\mu^5)\left(\frac{2}{x}\right)^{4/3}$$
$$+ e^{i\pi/3}L_{52}(\mu^2 - 2\mu^4 + \mu^6)\left(\frac{2}{x}\right)^{5/3} + (L_{61}\mu + L_{63}\mu^3 + L_{65}\mu^5 + L_{67}\mu^7)\left(\frac{2}{x}\right)^2 + O(x^{-7/3}), \tag{3a}$$

where

$$\mu = (1 - N^2)^{-1/2}, \tag{3b}$$

the coefficients are

$$
\begin{aligned}
L_{10} &= \frac{\eta_0^2}{60}, & L_{30} &= -\frac{\eta_0^3}{1\,400} - \frac{1}{140}, \\
L_{50} &= -\frac{281\eta_0^4}{4\,536\,000} - \frac{29\eta_0}{12\,600}, & L_{21} &= \frac{\eta_0}{6}, \\
L_{41} &= \frac{\eta_0^2}{360}, & L_{43} &= \frac{13\eta_0^2}{180}, \\
L_{45} &= -\frac{3\eta_0^2}{40}, & L_{52} &= \frac{\eta_0}{36}, \\
L_{61} &= -\frac{\eta_0^3}{2\,800} - \frac{1}{280}, & L_{63} &= \frac{23\eta_0^3}{2\,800} - \frac{13}{1\,120}, \\
L_{65} &= -\frac{21\eta_0^3}{400} + \frac{3}{80}, & L_{67} &= \frac{5\eta_0^3}{112} - \frac{5}{224},
\end{aligned}
\tag{3c}
$$

and $Ai(\eta_0) = 0$.

These roots are located in regions 1 and 3, near the solid curve separating 2 and 3. The solutions ν_x, of $H_\nu^{(1)}(x) = 0$ fall on this curve and are given by the first part of (3a), which does not depend on N. The second part approaches -1 as $N \to 0$ so that $\nu = \nu_x - 1$ at this limit. As is shown in Ref. [3], the solution is correct only if

$$(1 - N^2)\left(\frac{x}{2}\right)^{2/3} > |\eta_0|,$$

i.e. N and η_0 determine the minimum value of x which may be used in (3a).

In region 4, the same procedure yields

$$\frac{Ai'(q)}{Ai(q)} = \Gamma_2,$$

where

$$
\begin{aligned}
\Gamma_2 = \frac{i}{\partial_\nu q}\Bigg[\left(\frac{1}{N^2} - \frac{\nu^2}{y^2}\right)^{1/2} &+ \frac{ix}{2N(x^2 - \nu^2)} - \frac{i}{3y} + \frac{i\partial_\nu p}{p} \\
&+ \frac{x}{8N(x^2 - \nu^2)^{3/2}}\left(1 - \frac{5\nu^2}{\nu^2 - x^2}\right) + O(x^{-3})\Bigg].
\end{aligned}
$$

The asymptotic roots of this equation are:

$$
\begin{aligned}
\nu = y + \eta_0\left(\frac{y}{2}\right)^{1/3} &+ L_{10}\left(\frac{2}{y}\right)^{1/3} + L_{30}\left(\frac{2}{y}\right) - L_{50}\left(\frac{2}{y}\right)^{5/3} + i\beta + iL_{21}(\beta + \beta^3)\left(\frac{2}{y}\right)^{2/3} \\
&- i(L_{41}\beta - L_{43}\beta^3 + L_{45}\beta^5)\left(\frac{2}{y}\right)^{4/3} + L_{52}(\beta^2 + 2\beta^4 + \beta^6)\left(\frac{2}{y}\right)^{5/3} \\
&- i(L_{61}\beta - L_{63}\beta^3 + L_{65}\beta^5 - L_{67}\beta^7)\left(\frac{2}{y}\right)^2 + O(y^{-7/3}),
\end{aligned}
\tag{4a}
$$

where

$$\beta = (1/N^2 - 1)^{-1/2}, \tag{4b}$$

and the coefficients are given by (3c). The first part of (4a), which does not depend on N, corresponds to the roots of $J_\nu(y) = 0$. These are located on the real ν axis with $\mathrm{Re}\ (\nu) < y$. The second part is positive and imaginary.

A different solution of (1) is required in regions 5 and 7 since

$$\left(\frac{1}{N^2} - 1\right)\left(\frac{y}{2}\right)^{2/3} > |\eta_0|$$

is not satisfied there (see [3]). For $y > \mathrm{Re}\ (\nu) > -y$ and $|\nu - y| = O(y)$, we have for the first term of (1)

$$x\,\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} = i(x^2 - \nu^2)^{1/2} - \frac{x^2}{2(x^2 - \nu^2)} + O(x^{-1}). \tag{5}$$

The second term may be written

$$y\,\frac{J_\nu'(y)}{J_\nu(y)} = -\frac{y^2}{2(y^2 - \nu^2)} - (y^2 - \nu^2)^{1/2}\left\{\sin W - \frac{1}{(y^2 - \nu^2)^{1/2}}\left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2 - y^2)}\right]\cos W\right.$$

$$\left. + O(y^{-2})\right\}\cdot\left\{\cos W + \frac{1}{(y^2 - \nu^2)^{1/2}}\left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2 - y^2)}\right]\sin W + O(y^{-2})\right\}^{-1}, \tag{6a}$$

where

$$W = (y^2 - \nu^2)^{1/2} - \nu\cos^{-1}(\nu/y) - \pi/4. \tag{6b}$$

After substitution in (1) it can be shown that

$$\tan W = U, \tag{7a}$$

where

$$U = -\left\{i\left(\frac{x^2 - \nu^2}{y^2 - \nu^2}\right)^{1/2} + \frac{\nu^2(x^2 - y^2)}{2(x^2 - \nu^2)(y^2 - \nu^2)^{3/2}}\right.$$

$$\left. + \frac{x^2 - y^2}{(y^2 - \nu^2)^{3/2}}\left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2 - y^2)}\right] + O(x^{-2})\right\}. \tag{7b}$$

Equations (7a) and (7b) can be solved by iteration after rewriting the former as:

$$W = \tan^{-1} U = n\pi + \frac{\pi}{2} - \frac{1}{U} + \frac{1}{3U^2} - \frac{1}{5U^5} + \frac{1}{7U^7} + \cdots. \tag{8}$$

First, the zero-order solution is found graphically (see Fig. 2) or numerically by solving the truncated equation

$$W|_{\nu_0 = \alpha_0 y} = n\pi + \pi/2, \qquad n = 0, 1, 2, \cdots,$$

where we have set $\nu_0 = \alpha_0 y + O(1)$ in (6b). After α_0 has been determined, ν_0 is substituted in (7b) to give $U = U_0 + O(y^{-1})$. Then $\nu_1 = \alpha_0 y + \alpha_1 + O(y^{-1})$ is computed by a repetition of the procedure. The result is

$$\nu = \alpha_0 y + i\,\frac{\tanh^{-1}\Omega}{\cos^{-1}\alpha_0} + \left\{\frac{U_1\Omega^2}{(1 - \Omega^2)\cos^{-1}\alpha_0} - \frac{(\tanh^{-1}\Omega)^2}{2(1 - \alpha_0^2)^{1/2}(\cos^{-1}\alpha_0)^3}\right\}\frac{1}{y} + O(y^{-2}), \tag{9a}$$

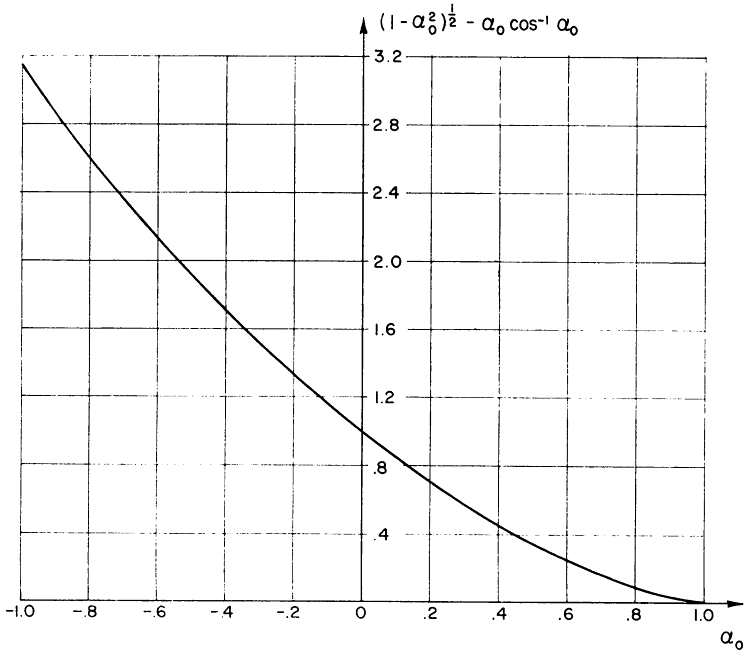

FIG. 2. $(1 - \alpha_0^2)^{1/2} - \alpha_0 \cos^{-1} \alpha_0$ vs. α_0 for the solution of $(1 - \alpha_0^2)^{1/2} - \alpha_0 \cos^{-1} \alpha_0 = (n\pi + 3\pi/4)/y$.

where

$$\Omega = N\left(\frac{1 - \alpha_0^2}{1 - \alpha_0^2 N^2}\right)^{1/2}, \tag{9b}$$

$$U_1 = -\frac{(1 - N^2)}{(1 - \alpha_0^2)^{3/2}}\left\{\frac{\alpha_0^2}{2(1 - \alpha_0^2 N^2)} + \frac{1}{8} + \frac{5\alpha_0^2}{24(1 - \alpha_0^2)} - \frac{\alpha_0 \tanh^{-1} \Omega}{N(1 - \alpha_0^2 N^2)^{1/2} \cos^{-1} \alpha_0}\right\}. \tag{9c}$$

All the roots given by (9a) are in the upper half ν plane. Their imaginary part, $\tanh^{-1} \Omega/\cos^{-1} \alpha_0$, is a monotonically increasing function of α_0 and N. Thus, for fixed N, successive roots approach the real axis as α_0 decreases toward -1. Equation (9a) is no longer accurate as $\alpha_0 \to -1$ and another solution must be sought in region 7. Also as $N \to 0$ the expressions for U given by (7b), which assumes $N = O(1) \neq 0$, is unbounded and (9a) becomes

$$\nu = \alpha_0 y + O(y^{-1}).$$

In this case it is convenient to replace ν with $-\nu$ in equation (1) and to solve

$$x\,\frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - y\,\frac{J'_{-\nu}(y)}{J_{-\nu}(y)} = 0 \tag{10}$$

in region 1. Here

$$J - \nu(y) = \tfrac{1}{2}[H_{-\nu}^{(1)}(y) + H_{-\nu}^{(2)}(y)] = \tfrac{1}{2}[e^{i\nu\pi}H_\nu^{(1)}(y) + e^{-i\nu\pi}H_\nu^{(2)}(y)]$$

$$= \left(\frac{2}{y}\right)^{1/3} p(y, \nu)\{\cos \nu\pi Ai(q) + \sin \nu\pi Bi(q)\},$$

$$y\,\frac{J'_{-\nu}(y)}{J_{-\nu}(y)} = -\frac{1}{3} + y\,\frac{\partial_\nu p}{p} + y\partial_\nu q\left\{\frac{\cos \nu\pi Ai'(q) + \sin \nu\pi Bi'(q)}{\cos \nu\pi Ai(q) + \sin \nu\pi Bi(q)}\right\}.$$

Substituting this expression and the appropriate formula for $xH_\nu^{(1)\prime}(x)/H_\nu^{(1)}(x)$ in equation (10) yields

$$\frac{\cos \nu\pi Ai(q) + \sin \nu\pi Bi(q)}{\cos \nu\pi Ai'(q) + \sin \nu\pi Bi'(q)} = \phi \tag{11a}$$

where

$$\phi = -i\partial_\nu q \left\{ \left(\frac{1}{N^2} - \frac{\nu^2}{y^2} \right)^{1/2} + \frac{ix}{2N(x^2 - \nu^2)} - \frac{1}{3y} + i\frac{\partial_\nu p}{p} + O(x^{-2}) \right\}^{-1}. \tag{11b}$$

Since $|\phi|$ is small when N is small, it is reasonable to begin by expanding numerator and denominator of (11a) about $\nu = \nu_0$ where

$$\cos \nu_0\pi Ai[q(\nu_0)] + \sin \nu_0\pi Bi[q(\nu_0)] = 0. \tag{12}$$

This process is greatly simplified by defining the functions

$$A(\nu) = \cos \nu\pi Ai(q) + \sin \nu\pi Bi(q), \qquad B(\nu) = \cos \nu\pi Ai'(q) + \sin \nu\pi Bi'(q),$$
$$C(\nu) = -\sin \nu\pi Ai(q) + \cos \nu\pi Bi(q), \quad D(\nu) = -\sin \nu\pi Ai'(q) + \cos \nu\pi Bi'(q).$$

Then equations (11a) and (12) become respectively

$$A(\nu) = \phi B(\nu) \tag{13}$$

and

$$A(\nu_0) = 0. \tag{14}$$

Expanding $A(\nu)$ and $B(\nu)$ in Taylor series about $\nu = \nu_0$ and simplifying the derivatives, we obtain

$$A(\nu) = (\pi C_0 + B_0 q_{\nu_0})\delta\nu + 2\pi D_0 q_{\nu_0} \frac{(\delta\nu)^2}{2!} - \pi^3 C_0 \frac{(\delta\nu)^3}{3!} + O(y^{-4/3}) \tag{15a}$$

and

$$B(\nu) = B_0 + \pi D_0 \delta\nu - \pi^2 B_0 \frac{(\delta\nu)^2}{2!} + O(y^{-1}), \tag{15b}$$

where $\delta\nu = \nu - \nu_0$, $q_{\nu_0} = \partial_\nu q\,|_0$, and the subscript denotes evaluation at $\nu = \nu_0$. Now if $\delta\nu$ has an asymptotic expansion of the form

$$a_0\phi + a_1\phi^2 + a_2\phi^3 + O(y^{-4/3}),$$

where $\phi = O(y^{-1/3})$, then a_0, a_1, and a_2 can be computed by equating the coefficients of equal orders in (13). Thus

$$\begin{aligned} \nu = \nu_0 &+ ia_0 \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1/2} \left(\frac{2}{y} \right)^{1/3} \\ &- \left\{ \frac{D_0}{C_0} \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1/2} + ia_0 \right\} a_0 \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1/2} \left(\frac{2}{y} \right)^{2/3} \\ &+ \left\{ \frac{2}{15} i(\nu_0 - y) \left(\frac{2}{y} \right)^{1/3} - i\left(\frac{D_0}{C_0} \right)^2 \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1} + \frac{3D_0}{C_0} a_0 \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1/2} \right. \\ &\left. + \frac{i\pi^2}{3} \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1} a_0^2 + ia_0^2 \right\} a_0 \left(\frac{1}{N^2} - \frac{\nu_0^2}{y^2} \right)^{-1/2} \left(\frac{2}{y} \right) + O(y^{-4/3}) \end{aligned} \tag{16a}$$

where

$$a_0 = \frac{B_0}{\pi C_0}. \tag{16b}$$

The quantities B_0, C_0, and D_0 are found in terms of $Ai(q_0)$ and $Ai'(q_0)$ by means of the Wronskian relation

$$Ai(q)Bi'(q) - Ai'(q)Bi(q) = \frac{1}{\pi}$$

and equation (14). The important results are

$$\frac{D_0}{C_0} = \frac{Ai'(q_0)}{Ai(q_0)} - \frac{\sin 2\nu_0\pi}{2\pi[Ai(q_0)]^2},$$

$$a_0 = -\left[\frac{\sin \nu_0\pi}{\pi Ai(q_0)}\right]^2 = -\left[\frac{\cos \nu_0\pi}{\pi Bi(q_0)}\right]^2.$$

Since a_0 is negative for all real ν_0, the solutions of equation (10) described by (16a) have negative imaginary parts and those of (1) have positive imaginary parts. We note that $\mathrm{Im}\,(\nu) = O(y^{-1/3})$ here, where previously it was $O(1)$.

In order to obtain numerical values for ν_0, equation (12) is rewritten, for $\cos \nu_0\pi \neq 0$ and $Bi(q_0) \neq 0$ as

$$\frac{Ai(q_0)}{Bi(q_0)} + \tan \nu_0\pi = \tan \chi(q_0) + \tan \nu_0\pi = 0.$$

This implies $\tan\,[\chi(q_0) + \nu_0\pi] = 0$ or

$$\nu_0\pi + \chi(q_0) = n\pi \qquad n = \pm\, 0, 1, 2 \cdots . \tag{17}$$

The function $\chi(q_0)$ is tabulated in Ref. [5], and the left side of (17) may be plotted as a function of ν_0 to obtain numerical solutions. Such a plot is given in Fig. 3 for $y = 16$.

This result ceases to be accurate when $|\nu_0 + y| = O(y)$ because the series for q diverges. For this case with $\mathrm{Re}\,(\nu) < -y$, Eq. (1) becomes (see the Appendix)

$$\frac{K_0(\nu)}{K_1(\nu)} = -(\nu^2 - y^2)^{-1/2} x \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} + O(y^{-1}), \tag{18a}$$

where

$$K_{0,1}(\nu) = 2 \pm \cot \nu\pi \exp(-2Z)[1 + O(y^{-1})], \tag{18b}$$

$$Z = -(\nu^2 - y^2)^{1/2} - \nu \cosh^{-1}(-\nu/y), \tag{18c}$$

and the lower sign in (18b) refers to $K_1(\nu)$. As ν varies from $-y$ to $-\infty$ along the negative real axis, the right hand side of (18a) changes in the fourth quadrant from $-i\infty$ to $+1$, approaching $+1$ from below as shown in Fig. 4. Since Z is a large positive monotonically increasing function of $-\nu$ in this range, $\exp(-2Z)$ is exceedingly small. Thus for (18a) to be an equality ν must be near solutions of $(\cot \nu\pi)^{-1} = 0$, i.e., the negative real integers. Setting $\nu\pi = m\pi + \delta$, $|\delta| \ll 1$ we obtain

$$\frac{K_0(\nu)}{K_1(\nu)} \approx \frac{1 + e^{-2Z}/(2\delta)}{1 - e^{-2Z}/(2\delta)}$$

and the equality of (18a) requires

$$|\delta| = O[\exp(-2Z)] \quad \text{and} \quad \mathrm{Im}\,(\delta) > 0.$$

The detailed calculation of δ is unnecessary.

Numerical results computed from equations (4a), (9a), and the negative of (16a) are plotted in Fig. 5 for $y = 16$. The roots with $\mathrm{Re}\,(\nu) \approx 11.5$ and $\mathrm{Re}\,(\nu) \approx -13.5$ illustrate the close agreement between the alternate representations in the regions where

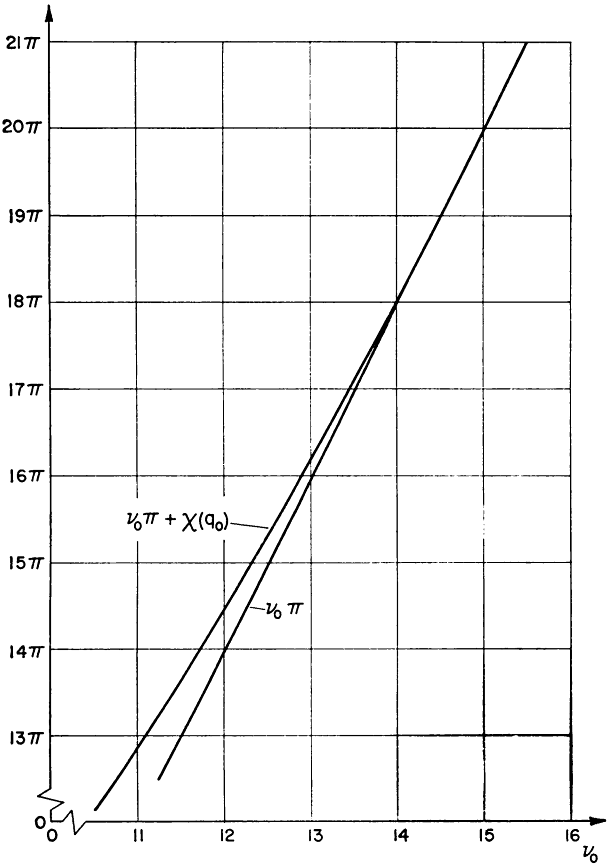

FIG. 3. $\nu_0 \pi + \chi(q_0)$ vs. ν_0 for $y = 16$.

they overlap. These coincident solutions determine the regions where the various formulae apply.

It remains to compute the roots in the lower half plane. Since these correspond to the solutions of $H_\nu^{(1)}(x) = 0$, we examine the substitution $2J_\nu(y) \sim H_\nu^{(2)}(y)$, which applies for Im $(\nu) < -1$ (see the Appendix). Then (1) becomes

$$x \frac{H_\nu^{(1)\prime}(x)}{H_\nu^{(1)}(x)} - y \frac{H_\nu^{(2)\prime}(y)}{H_\nu^{(2)}(y)} = 0, \tag{19}$$

which was treated in Ref. [3]. The roots in region 10 are therefore given by the negative of

$$\begin{aligned}\nu = x - \eta_0 e^{i\pi/3} \left(\frac{x}{2}\right)^{1/3} + e^{i2\pi/3} L_{10} \left(\frac{2}{x}\right)^{1/3} + L_{30} \left(\frac{2}{x}\right) + e^{i\pi/3} L_{50} \left(\frac{2}{x}\right)^{5/3} \\ + \mu - e^{i\pi/3} L_{21}(\mu - \mu^3) \left(\frac{2}{x}\right)^{2/3} - e^{i2\pi/3}(L_{41}\mu + L_{43}\mu^3 + L_{45}\mu^5) \left(\frac{2}{x}\right)^{4/3} \\ + e^{i\pi/3} L_{52}(\mu^2 - 2\mu^4 + \mu^6) \left(\frac{2}{x}\right)^{5/3} - (L_{61}\mu + L_{63}\mu^3 + L_{65}\mu^5 + L_{67}\mu^7) \left(\frac{2}{x}\right)^2 + O(x^{-7/3}),\end{aligned} \tag{20}$$

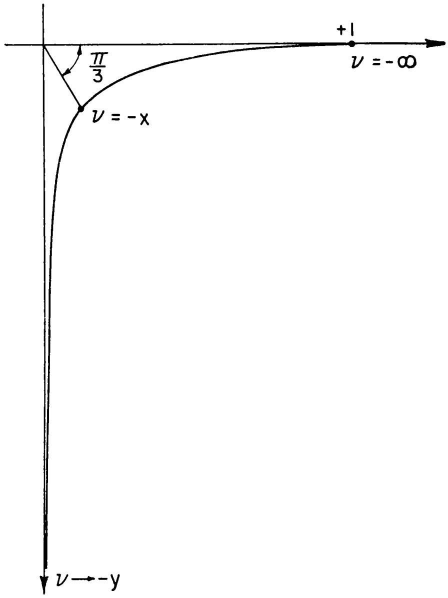

Fig. 4. The locus of $-(\nu^2 - y^2)^{-1/2}\, x(H_\nu^{(1)\prime}(x)/H_\nu^{(1)}(x))$ for $-\infty \leq \nu \leq -y$.

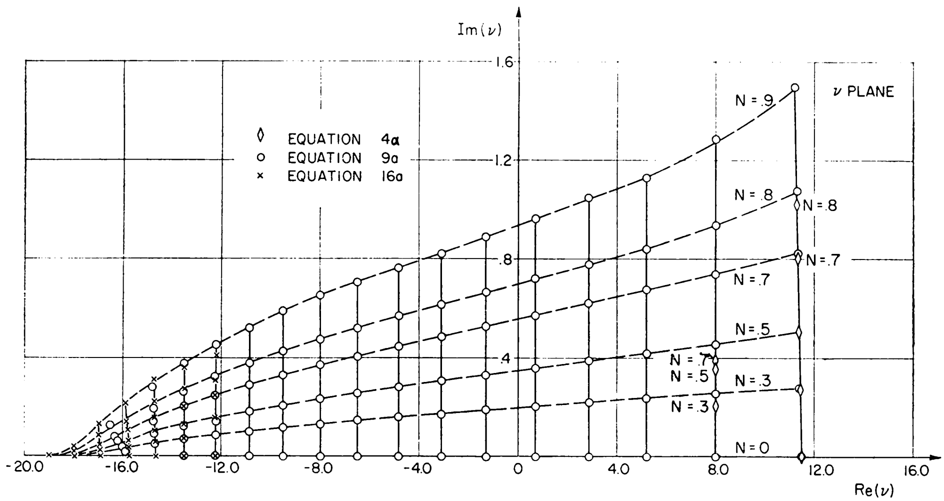

Fig. 5. Roots of $x(H_\nu^{(1)\prime}(x)/H_\nu^{(1)}(x)) - y(J'\nu(y)/J\nu(y)) = 0$ for $y = 16$, $N = y/x$.

where the abbreviations of (3b) and (3c) have been employed. Since the imaginary part of (20) for the smallest value of $|\eta_0|$ is approximately $-2(x/2)^{1/3}$, the use of equation (19) in lieu of (1) is justified. Plots of equation (20) are given in Ref. [3].

Appendix—Asymptotic Properties of $J_\nu(y)$

The required properties of the Bessel and Hankel functions are discussed by Franz and Beckmann [4] and Schöbe [6], while the Airy functions are considered by Miller [5]. Reference [3] contains a summary of their results.

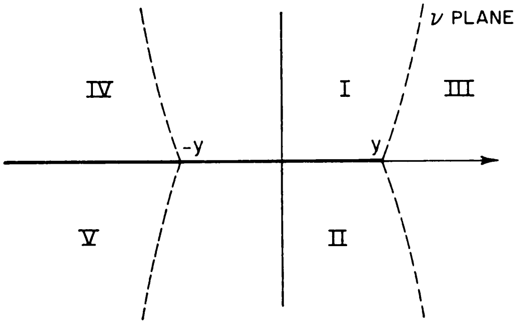

FIG. 6. Regions in the ν-plane for $J\nu(y)$.

Franz and Beckmann have shown that the solutions of $H_\nu^{(1)}(y) = 0$ for real y lie in the first and third quadrants of the ν-plane on curves symmetric through the origin. The curve in the first quadrant is approximately given by

$$\text{Im}\ [(y^2 - \nu^2)^{1/2} - \nu \cos^{-1} (\nu/y)] = 0,$$

which is the solid curve separating regions 2 and 3 in Fig. 1. The asymptotic formula for $H_\nu^{(1)}(x)$ is a continuous function of ν so long as these curves are not crossed. The same is true of $H_\nu^{(2)}(y)$, whose zeros lie on the corresponding curves in the second and fourth quadrants. The roots of $J_\nu(y) = 0$ fall on the real ν axis with $\nu < y$. Except for this line, the asymptotic formula for $J_\nu(y)$ is a continuous function of ν. The asymptotic formulas for $J_\nu(y)$ are most conveniently expressed by dividing the ν plane into regions as shown in Fig. 6.

The asymptotic form in region III for $|\nu - y| = O(y)$ is

$$J_\nu(y) = (2\pi)^{-1/2} \frac{\exp\ [(\nu^2 - y^2)^{1/2} - \nu \cosh^{-1} (\nu/y)]}{(\nu^2 - y^2)^{1/4}} \cdot \left\{1 + \frac{1}{(\nu^2 - y^2)^{1/2}} \left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2 - y^2)}\right] + O(y^{-2})\right\}, \tag{A-1}$$

where

$$|\arg\ (\nu^2 - y^2)| < \pi/2, \quad \text{Re}\ [\cosh^{-1} (\nu/y)] > 0, \quad |\text{Im}\ [\cosh^{-1} (\nu/y)]| < \pi/2.$$

In all the other regions the use of $J_\nu(y) = \frac{1}{2}[H_\nu^{(1)}(y) + H_\nu^{(2)}(y)]$ yields the correct results. For regions I and II near the real axis with $|\nu \pm y| = O(y)$, we have

$$J_\nu(y) = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(y^2 - \nu^2)^{1/4}} \left\{\cos W + \frac{1}{(y^2 - \nu^2)^{1/2}} \left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2 - y^2)}\right] \sin W + O(y^{-2})\right\}, \tag{A-2a}$$

where

$$W = (y^2 - \nu^2)^{1/2} - \nu \cos^{-1} (\nu/y) - \pi/4, \tag{A-2b}$$

$$|\arg\ (y^2 - \nu^2)^{1/2}| < \pi/2 \qquad 0 \le \text{Re}\ [\cos^{-1} (\nu/y)] \le \pi.$$

The exponential contributed by $H_\nu^{(1)}(y)$ becomes dominant in region I, while in region II,

$H_\nu^{(2)}(y)$ is dominant. Thus far from the real axis we have

$$J_\nu(y) = \frac{(2\pi)^{-1/2}}{(y^2-\nu^2)^{1/4}} \exp(\pm iW) \left\{1 \mp \frac{i}{(y^2-\nu^2)^{1/2}}\left[\frac{1}{8} - \frac{5\nu^2}{24(\nu^2-y^2)}\right] + O(y^{-2})\right\}, \quad \text{(A-3)}$$

where the upper signs refer to region I.

In regions IV and V near the real ν axis with $|\nu + y| = O(y)$ we have

$$\begin{aligned} J_\nu(y) &= \tfrac{1}{2}[H_\nu^{(1)}(y) + H_\nu^{(2)}(y)] = \tfrac{1}{2}[e^{-i\nu\pi}H_{-\nu}^{(1)}(y) + e^{i\nu\pi}H_{-\nu}^{(2)}(y)] \\ &= -\frac{(2\pi)^{-1/2}}{(\nu^2-y^2)^{1/4}}\left\{2\sin\nu\pi\exp(Z)\left[1 - \frac{1}{(\nu^2-y^2)^{1/2}}\left(\frac{1}{8} - \frac{5\nu^2}{24(\nu^2-y^2)}\right) + O(y^{-2})\right]\right. \\ &\qquad\left. - \cos\nu\pi\exp(-Z)[1 + O(y^{-1})]\right\} \end{aligned} \quad \text{(A-4a)}$$

where

$$Z = -(\nu^2 - y^2)^{1/2} - \nu\cosh^{-1}(-\nu/y), \quad \text{(A-4b)}$$

and

$$|\arg(\nu^2-y^2)^{1/2}| < \pi/2, \quad \operatorname{Re}[\cosh^{-1}(-\nu/y)] > 0, \quad |\operatorname{Im}[\cosh^{-1}(-\nu/y)]| < \pi/2.$$

Here the very small exponential term $\exp(-Z)$, which is only important at the solutions of $\sin\nu\pi = 0$, has been included. At points other than the negative integers, $\exp(-Z)$ is ignored and $H_{-\nu}^{(1)}(y) \sim -H_{-\nu}^{(2)}(y)$ or

$$|H_\nu^{(1)}(y)/H_\nu^{(2)}(y)| = |\exp(-2\pi i\nu)| = \exp[2\pi \operatorname{Im}(\nu)].$$

Thus in region IV

$$J_\nu(y) \sim \tfrac{1}{2}H_\nu^{(1)}(y)$$

and in region V

$$J_\nu(y) \sim \tfrac{1}{2}H_\nu^{(2)}(y).$$

When $|\nu - y| = O(y^{1/3})$ we have [6]

$$J_\nu(y) = \left(\frac{2}{y}\right)^{1/3} p(y,\nu)Ai[q(y,\nu)] + O[y^{-2/3(m+1)}], \quad \text{(A-5)}$$

where

$$p(y,\nu) = \sum_{j=0}^{m}(-1)^j P_j(\xi)\left(\frac{2}{y}\right)^{2j/3}, \quad \text{(A-6a)}$$

$$q(y,\nu) = \sum_{k=0}^{m}(-1)^k Q_k(\xi)\left(\frac{2}{y}\right)^{2k/3}, \quad \text{(A-6b)}$$

$$\xi = (\nu - y)\left(\frac{2}{y}\right)^{1/3},$$

$$P_0(\xi) = 1, \qquad Q_0(\xi) = \xi,$$

$$P_1(\xi) = \frac{1}{15}\xi, \qquad Q_1(\xi) = \frac{1}{60}\xi^2,$$

$$P_2(\xi) = \frac{13}{1\,260}\xi^2, \qquad Q_2(\xi) = \frac{2}{1\,575}\xi^3 + \frac{1}{140},$$

$$P_3(\xi) = \frac{109}{56\,700}\xi^3 + \frac{1}{900}, \qquad Q_3(\xi) = \frac{41}{283\,500}\xi^4 + \frac{4}{1\,575}\xi.$$

When $|\nu + y| = O(y^{1/3})$ relationships such as $J_\nu(y) = \frac{1}{2}[e^{-i\nu\pi}H_{-\nu}^{(1)}(y) + e^{i\nu\pi}H_{-\nu}^{(2)}(y)]$ will permit the use of formulae which apply for $|\nu - y| = O(y^{1/3})$.

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