ON THE TIME DEPENDENCE OF VISCOELASTIC VARIATIONAL SOLUTIONS*

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Abstract. Thermodynamic operational-variational principles are employed in a study of the transient response of linear viscoelastic media with an arbitrary degree of anisotropy. Assuming displacements in the form of a series of products of space-dependent functions and time-dependent generalized coordinates, the (approximate) response is calculated by minimizing a functional which is analogous to the potential energy of an elastic body. Similarly, a principle analogous to the principle of minimum complementary energy of elasticity is used to deduce transient behavior of (approximate) stresses. The displacements are not required to satisfy equilibrium or stress boundary conditions, nor are stresses calculated from the complementary principle required to satisfy compatibility or displacement boundary conditions. It is found that when applied loads and displacements are step-functions of time, the transient component of stresses and displacements is given in most cases by a series of exponentials with negative, real arguments.

1. Introduction. In a recent paper [1] variational principles in *linear* thermoviscoelasticity were deduced from Biot's thermodynamic theory [2]. Anticipating various applications, these principles were stated using convolution-type functionals expressed in terms of several different combinations of mechanical displacements, entropy displacements (defined as the heat flow vector divided by a reference temperature), stresses, and temperature. Special cases of these general principles are Biot's operational-variational principles for mechanical displacements and/or entropy displacements in isothermal viscoelasticity [3] and thermoelasticity [4], and Gurtin's [5] convolution principles for mechanical displacements and/or stresses in isothermal viscoelasticity.

Assuming zero initial conditions, all of the convolution functionals [1, 5] have the characteristic form

$$I = \int_V f_i^* g_i \, dV + \int_A F_i^* G_i \, dA,$$

where f_i, g_i, F_i, and G_i are functions of spacial coordinates and time, integration is over the time-independent volume V and bounding surface A, the usual tensor notation is employed in that repetition of the index implies summation over its range, and the asterisk between two functions is shorthand for convolution integration with respect to time; e.g.,

$$f_i^* g_i \equiv \int_0^t f_i(\tau) g_i(t - \tau) \, d\tau.$$

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When we take the Laplace transform of I, viz.

$$I'(p) = \int_0^\infty e^{-pt} I(t)\, dt,$$

an operational form of the functional is obtained, according to the familiar rule for the transform of convolution products [6],

$$I'(p) = \int_V f_i' g_i'\, dV + \int_A F_i' G_i'\, dA,$$

in which a prime designates the Laplace transform of the function. By making $I(t)$ stationary with respect to a certain class of variations of the dependent variables, integro-differential Euler equations and natural boundary conditions for the space- and time-dependent variables are obtained. On the other hand, by applying the condition for stationary behavior to the operational form $I'(p)$, we obtain the Laplace transform of these Euler equations and boundary conditions. Principles utilizing $I'(p)$ are called operational-variational principles.

The purpose of this present report is to derive the form of time dependence of approximate (or possibly exact) viscoelastic solutions that are obtained by applying the Ritz method [7] to certain of the transformed functionals. For algebraic simplicity we have omitted all thermal variables here, and consider only a principle for mechanical displacements and a complementary principle for stresses. However, the results are readily extended to the thermoviscoelastic principles in [1] that involve functionals whose Laplace transform on the positive, real axis of the transform parameter, p, attains an absolute minimum for the exact solution. The principles in [1] that satisfy this positive property are the two "homogeneous principles" (one for mechanical and entropy displacements and one for mechanical stresses and temperature) in which the thermal variables may or may not be thermodynamically coupled with the mechanical variables, and two "nonhomogeneous principles" (one in terms of mechanical displacements and temperature and one in terms of mechanical stresses and entropy displacements) in which either the mechanical or thermal variables must be prescribed throughout the body. Minimum principles for isothermal viscoelastic behavior and heat conduction, for example, are special cases of these four principles.

The following analyses apply to linear media with an arbitrary degree of anisotropy. Also, it is assumed that all prescribed loads and displacements are step-functions of time applied at $t = 0$; generalization can, of course, be accomplished by using superposition.

2. Displacement response. In this section an operational-variational principle, which is similar to the well-known principle of minimum potential energy for elastic bodies [7], will be used to calculate time dependence of the displacement vector field (u_1, u_2, u_3) given by the series

$$u_i = f_i^\alpha(x) q_\alpha(t) + U_i(x) H(t), \qquad (i = 1, 2, 3). \tag{1}$$

Summation over $\alpha(=1, 2, \cdots, N)$ is implied, $f_i^\alpha(x)$ are assumed functions of only the coordinates x_i and vanish on the portion of the boundary A_u where displacements are prescribed, $q_\alpha(t)$ are unspecified time-dependent generalized coordinates, and $U_i(x)H(t)$ is the prescribed displacement vector for which $H(t)$ is the Heaviside step-function

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{2}$$

It is not required that these displacements satisfy equilibrium or stress boundary conditions.

The appropriate displacement functional which is to be minimized for calculation of the generalized coordinates is [1]

$$I'_u = \frac{1}{2} \int_V Z^{kl}_{ij} e'_{ij} e'_{kl} \, dV - \int_V F'_{iH} u'_i \, dV - \int_{A_T} T'_{iH} u'_i \, dA, \qquad (i, j, k, l = 1, 2, 3). \tag{3}$$

where, as before, repetition of the indices in each product implies summation over their range. T'_{iH} and F'_{iH} are prescribed, transformed forces given by

$$T'_{iH} = T_i(x)/p, \qquad F'_{iH} = F_i(x)/p. \tag{4}$$

A_T is the portion of the boundary where forces are prescribed, the e'_{ij} are transformed strains, and the transformed displacement vector is

$$u'_i = f^\alpha_i q'_\alpha + U_i/p. \tag{5}$$

Also, from thermodynamic studies for stable systems [1, 2], the operational moduli Z^{kl}_{ij} are given by

$$Z^{kl}_{ij} = \sum_s \frac{pD^{kl(s)}_{ij}}{p + (1/\rho_s)} + D^{kl}_{ij} + pD'^{kl}_{ij}, \tag{6}$$

where each matrix in (6) is completely symmetric, real, and positive semi-definite, i.e.

$$\begin{aligned} D^{kl(s)}_{ij} e_{ij} e_{kl} &\geq 0, \qquad D^{kl}_{ij} e_{ij} e_{kl} \geq 0, \\ D'^{kl}_{ij} e_{ij} e_{kl} &\geq 0, \qquad e_{ij} e_{ij} > 0, \end{aligned} \tag{7}$$

but the matrix made up of the sum of those in (6) is positive definite, i.e.

$$[\sum_s D^{kl(s)}_{ij} + D^{kl}_{ij} + D'^{kl}_{ij}] e_{ij} e_{kl} > 0, \qquad e_{ij} e_{ij} > 0. \tag{8}$$

Also, the "relaxation times," ρ_s, are real and positive.

We now write I'_u as a function of q'_α by using the transformed strain-displacement relations

$$e'_{ij} = \tfrac{1}{2}(u'_{i,j} + u'_{j,i}), \tag{9}$$

to find

$$\begin{aligned} I'_u = \frac{1}{2} \int_V Z^{kl}_{ij} [e^\alpha_{ij} q'_\alpha + E_{ij}/p][e^\beta_{kl} q'_\beta + E_{kl}/p] \, dV \\ - \int_V [f^\alpha_i q'_\alpha + U_i/p] \frac{F_i}{p} \, dV - \int_{A_T} [f^\alpha_i q'_\alpha + U_i/p] \frac{T_i}{p} \, dA, \end{aligned} \tag{10}$$

where

$$e^\alpha_{ij} \equiv \tfrac{1}{2}(f^\alpha_{i,j} + f^\alpha_{j,i}), \tag{11}$$

$$E_{ij} \equiv \tfrac{1}{2}(U_{i,j} + U_{j,i}). \tag{12}$$

The generalized coordinates are found by minimizing I'_u with respect to each q'_α which leads to the following N linear algebraic equations:

$$C_{\alpha\beta} q'_\alpha = \langle Q_\beta \rangle / p^2, \tag{13}$$

in which the following definitions are employed:

$$C_{\alpha\beta} \equiv \int_V \left[\sum_s \frac{D_{ij}^{kl(s)}}{p + (1/\rho_s)} + D_{ij}^{kl}/p + D_{ij}^{\prime\prime kl} \right] e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{14}$$

$$\langle Q_\beta \rangle \equiv \int_{A_T} T_i f_i^\beta \, dA + \int_V F_i f_i^\beta \, dV - \int_V \left[\sum_s \frac{p D_{ij}^{kl(s)}}{p + (1/\rho_s)} + D_{ij}^{kl} + p D_{ij}^{\prime\prime kl} \right] e_{ij}^\beta E_{kl} \, dV. \tag{15}$$

It is noted that the singularities of $\langle Q_\beta \rangle$ are simple poles (or branch cuts if $\rho_s = \rho_s(x_i)$) on the negative real p-axis, but if the boundary conditions are all on stress $\langle Q_\beta \rangle$ is independent of p. Also, from the symmetry of Z_{ij}^{kl} we have $C_{\alpha\beta} = C_{\beta\alpha}$. The operational moduli in (14) and (15) have been left in the volume integrals since properties may be functions of x_i. We shall now establish the dependence of q'_α on p, and thereby obtain the time dependence of displacements.

First, the following theorem will be proved:

THEOREM I. *The singularities of q'_α occur only on the nonpositive real p-axis.*

The proof will be made by showing that the determinant of $C_{\alpha\beta}$ (denoted by $|C_{\alpha\beta}|$) does not vanish when p is complex or real and positive. Let $p = u + iv$ and substitute this into (14) to find

$$C_{\alpha\beta} = R_{\alpha\beta} - ivI_{\alpha\beta}, \tag{16}$$

where

$$R \equiv \int_V \left[\sum_s \frac{[u + (1/\rho_s)] D_{ij}^{kl(s)}}{[u + (1/\rho_s)]^2 + v^2} + \frac{u D_{ij}^{kl}}{u^2 + v^2} + D_{ij}^{\prime\prime kl} \right] e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{17}$$

$$I_{\alpha\beta} \equiv \int_V \left[\sum_s \frac{D_{ij}^{kl(s)}}{[u + (1/\rho_s)]^2 + v^2} + \frac{D_{ij}^{kl}}{u^2 + v^2} \right] e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{18}$$

and

$$I_{\alpha\beta} = I_{\beta\alpha}, \qquad R_{\alpha\beta} = R_{\beta\alpha}. \tag{19}$$

It is noted by reference to equations (6)–(8) that $R_{\alpha\beta}$ is positive definite when $u > 0$, but is indefinite when $u < 0$; also $I_{\alpha\beta}$ is positive semi-definite for all u and v.

Let us assume that $|C_{\alpha\beta}| = 0$ and determine the permissible values of u and v. Vanishing of this determinant means that a non-trivial (real or complex) solution y_α can be found such that

$$(R_{\alpha\beta} - ivI_{\alpha\beta}) y_\alpha = 0. \tag{20}$$

If the complex conjugate of y_β is denoted by y_β^c, then multiplying (20) by y_β^c and summing yields

$$R_{\alpha\beta} y_\alpha y_\beta^c = ivI_{\alpha\beta} y_\alpha y_\beta^c. \tag{21}$$

Since $R_{\alpha\beta}$ and $I_{\alpha\beta}$ are real, symmetric matrices, $R_{\alpha\beta} y_\alpha y_\beta^c$ and $I_{\alpha\beta} y_\alpha y_\beta^c$ are real numbers;

in addition, the latter one is non-negative.* First, assume that $I_{\alpha\beta}y_\alpha y_\beta^c > 0$. But (21) cannot be satisfied unless v is zero since the left-hand side is real while the right-hand side is imaginary. Now, suppose $I_{\alpha\beta}y_\alpha y_\beta^c = 0$, which can be seen from (18) to imply that

$$R_{\alpha\beta}y_\alpha y_\beta^c = \left[\int_V D_{ij}^{\prime\prime kl} e_{ij}^\alpha e_{kl}^\beta \, dV\right] y_\alpha y_\beta^c \, . \tag{22}$$

But (8) requires that this be a (nonzero) positive number and, therefore, (21) cannot be satisfied. Thus, the determinant of $C_{\alpha\beta}$ cannot vanish unless p is real.

It only remains to show that there are no zeros of $|C_{\alpha\beta}|$ on the positive real axis. That this is indeed the case follows immediately from the fact that $R_{\alpha\beta}$ is positive definite when $u > 0$. Theorem I is, therefore, proved for the most general stress-strain relations which are thermodynamically admissible.

Further information about the singularities of q_α' will now be obtained. However, in the following discussion we shall assume that the relaxation times, ρ_s, are independent of x_i. This assumption permits $C_{\alpha\beta}$ (defined by (14)) to be written as

$$C_{\alpha\beta} = \sum_s \frac{F_{\alpha\beta}^{(s)}}{p + (1/\rho_s)} + \frac{F_{\alpha\beta}}{p} + F_{\alpha\beta}^{\prime\prime} \, , \tag{23}$$

with the definitions

$$F_{\alpha\beta}^{(s)} \equiv \int_V D_{ij}^{kl(s)} e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{24}$$

$$F_{\alpha\beta} \equiv \int_V D_{ij}^{kl} e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{25}$$

$$F_{\alpha\beta}^{\prime\prime} \equiv \int_V D_{ij}^{\prime\prime kl} e_{ij}^\alpha e_{kl}^\beta \, dV, \tag{26}$$

and the equation for q_α' becomes

$$\left[\sum_s \frac{pF_{\alpha\beta}^{(s)}}{p + (1/\rho_s)} + F_{\alpha\beta} + pF_{\alpha\beta}^{\prime\prime}\right] q_\alpha' = \langle Q_\beta \rangle / p. \tag{27}$$

The following theorem will be proved:

Theorem II. *When the relaxation times, ρ_s, are constant the singularities of q_α' are simple poles except at the origin and at $-1/\rho_s$ where double poles may occur.*

For the present let us assume that $\langle Q_\beta \rangle$ is independent of p. Also, let $\alpha, \beta = 1, 2, \cdots, N$ and $s = 1, 2, \cdots, M$, so that we can write (27) as

$$G_{\alpha\beta} q_\alpha' = \frac{\langle Q_\beta \rangle}{p} \prod_{s=1}^{M} [p + (1/\rho_s)], \tag{28}$$

in which each element of $G_{\alpha\beta}$ is at most a polynomial of order $N(M + 1)$. Theorem I implies

$$|G_{\alpha\beta}| = G \prod_{r=1}^{R} [p + (1/\gamma_r)]^{m_r}, \tag{29}$$

*Note that $I_{\alpha\beta}y_\alpha y_\beta$, with y_α real, is non-negative as a consequence of (7); furthermore, this property implies that $I_{\alpha\beta}y_\alpha y_\beta^c$, with y complex, is non-negative.

where G and γ_r are real, positive constants, m_r is the multiplicity of the rth root ($p = -1/\gamma_r$), and $\sum_{r=1}^{R} m_r \leq N(M+1)$. It follows that q'_α can be expressed as a ratio of polynomials in p given by

$$q'_\alpha = \frac{\sum_t a^{(t)}_{\alpha\beta} p^t}{\prod_{r=1}^{R} [p + (1/\gamma_r)]^{m_r}} \frac{\langle Q_\beta \rangle}{p}, \qquad t = 1, 2, \cdots, \tag{30}$$

where the $a^{(t)}_{\alpha\beta}$ are independent of p. The ratio multiplying $\langle Q_\beta \rangle / p$ can be written as a sum of partial fractions if the order of the numerator is lower than that of the denominator; if they are of equal order then q'_α will contain an additional constant term multiplying $\langle Q_\beta \rangle / p$. That either of these conditions is always satisfied can be shown to follow from (27) by letting $p \to \infty$. If $|F''_{\alpha\beta}| > 0$ then (27) shows that q'_α must behave like $1/p^2$ as $p \to \infty$; hence

$$\frac{\sum_t a^{(t)}_{\alpha\beta} p^t}{\prod_{r=1}^{R} [p + (1/\gamma_r)]^{m_r}} \sim \frac{1}{p}, \qquad p \to \infty. \tag{31}$$

If, however, $|F''_{\alpha\beta}| = 0$ then

$$\frac{\sum_t a^{(t)}_{\alpha\beta} p^t}{\prod_{r=1}^{R} [p + (1/\gamma_r)]^{m_r}} \sim \text{constant}, \qquad p \to \infty. \tag{32}$$

Consequently, it is always possible to write q'_α as the following partial fraction series with coefficients $S^{(t)}_{\alpha\beta}$ and $S_{\alpha\beta}$:

$$q'_\alpha = \left[\sum_{r=1}^{R} \sum_{t=1}^{m_r} \frac{S^{(t)}_{\alpha\beta}}{[p + (1/\gamma_r)]^t} + S_{\alpha\beta} \right] \frac{\langle Q_\beta \rangle}{p}. \tag{33}$$

The order of the poles of q'_α can be determined by examining the behavior of q'_α in conjunction with (27) as p approaches the roots of $|G_{\alpha\beta}|$. Consider then $p = \epsilon - 1/\gamma_r$ and $|\epsilon| \ll 1$; with p close to $-1/\gamma_r$ only the term in (33) which behaves like ϵ^{-m_r} need be retained, thus

$$q'_\alpha \simeq \frac{S^{(m_r)}_{\alpha\beta}}{\epsilon^{m_r}} \frac{\langle Q_\beta \rangle}{p} \equiv \frac{g^{(r)}_\alpha}{p \epsilon^{m_r}}, \tag{34}$$

with $g^{(r)}_\alpha$ defined as $\langle Q_\beta \rangle S^{(m_r)}_{\alpha\beta}$. Multiplying (27) by q'_β and summing over β, and then substituting (34) for q'_α yields

$$\sum_s \frac{F^{(s)}}{p + (1/\rho_s)} + \frac{F}{p} + F'' = \epsilon^{m_r} g^{(r)}_\beta \frac{\langle Q_\beta \rangle}{p}, \tag{35}$$

where $p = \epsilon - 1/\gamma_r$ and

$$F^{(s)} \equiv F^{(s)}_{\alpha\beta} g^{(r)}_\alpha g^{(r)}_\beta \geq 0, \tag{36}$$

$$F \equiv F_{\alpha\beta} g^{(r)}_\alpha g^{(r)}_\beta \geq 0, \tag{37}$$

$$F'' \equiv F''_{\alpha\beta} g^{(r)}_\alpha g^{(r)}_\beta \geq 0, \tag{38}$$

in which (r) is *not* summed out. The right-hand side of (35) has a zero of order m_r, or greater if $g_\beta^{(r)}\langle Q_\beta \rangle = 0$, whose value must agree with the left-hand side. However, the latter can have only simple zeros at all poles of q'_α since its first derivative, which is

$$-\sum_s \frac{F^{(s)}}{[p + (1/\rho_s)]^2} - \frac{F}{p^2} < 0, \tag{39}$$

can never vanish in the finite p-plane; note that $\sum_s F^{(s)} + F > 0$ in order for a zero to exist in (35). Thus, $m_r \leq 1$ for all finite and infinite values of γ_r. It is to be noted that this conclusion applies even when a γ_r is equal to one of the relaxation times, ρ_s. Reference to (35) shows that this equality can occur only if $F^{(s)} = 0$. Also, (27) indicates that the determinant, $|G_{\alpha\beta}|$, has a zero at the origin if and only if $|F_{\alpha\beta}| = 0$. If this latter condition exists q'_α has a double pole at the origin.

Theorem II is thus proved for the case in which $\langle Q_\beta \rangle$ is constant. Furthermore, the above considerations, upon inverting the transforms, lead to the following time dependence of the generalized coordinates:

1. When $|F_{\alpha\beta}| > 0$, $|F''_{\alpha\beta}| > 0$:

$$q_\alpha = \langle Q_\beta \rangle \sum_r \gamma_r S_{\alpha\beta}^{(r)} (1 - e^{-t/\gamma_r}). \tag{40}$$

2. When $|F_{\alpha\beta}| > 0$, $|F''_{\alpha\beta}| = 0$:

$$q_\alpha = \langle Q_\beta \rangle [\sum_r \gamma_r S_{\alpha\beta}^{(r)} (1 - e^{-t/\gamma_r}) + S_{\alpha\beta}]. \tag{41}$$

3. When $|F_{\alpha\beta}| = |F''_{\alpha\beta}| = 0$:

$$q_\alpha = \langle Q_\beta \rangle [\sum_r \gamma_r S_{\alpha\beta}^{(r)} (1 - e^{-t/\gamma_r}) + S_{\alpha\beta} + S''_{\alpha\beta} t]. \tag{42}$$

It is also readily shown that symmetry and realness of the matrices in (27) imply that $S_{\alpha\beta}^{(r)}$, $S_{\alpha\beta}$, and $S''_{\alpha\beta}$ are symmetric and real.

When the restriction that the $\langle Q_\beta \rangle$ are constant (i.e., independent of p) is removed, then q'_α contains poles at $-1/\rho_s$ and at $-1/\gamma_r$. Furthermore, through considerations similar to those used to show $m_r \leq 1$, we find that all poles are simple if $D_{ij}^{kl(s)}$ is positive definite for all s; if this is not the case, double poles may occur at $-1/\rho_s$. $q(t)$, therefore, has time dependence similar to that shown in Eqs. (40)–(42) except there will be additional exponentials with time constants ρ_s, and possibly terms of the form $te^{-(t/\rho_s)}$. Also, the correspondence between the vanishing of a given determinant and the time dependence indicated in cases 1, 2, and 3 above will not necessarily be the same. For example, if T_i and F_i in (15) are zero and $D_{ij}^{kl} \equiv F_{\alpha\beta} \equiv 0$, then q'_α will contain at most a simple pole at the origin; hence, its inverse will not have the term proportional to time which is shown in (42).

3. Stress response. Viscoelastic stresses that are derived from a complementary operational-variational principle, which is analogous to the principle of minimum complementary energy for elastic media [7], will now be shown to have time dependence similar to that of the displacements discussed above. We consider stresses expressed by the series

$$\sigma_{ij} = f_{ij}^\alpha(x) Q_\alpha(t) + T_{ij}(x) H(t), \tag{43}$$

in which α is to be summed out ($\alpha = 1, 2, \cdots, N$), f_{ij}^α $(=f_{ji}^\alpha)$ are given functions of the coordinates x_i which vanish on A_T where stresses are prescribed, Q_α are time-dependent functions which we shall call generalized stresses, and on A_T the vector $T_{ij}n_jH$ is equal to the prescribed surface force, T_{iH}. It is further assumed that for each α the f_{ij}^α satisfy the equilibrium equations

$$f_{ij,j}^\alpha = 0, \tag{44}$$

and the stresses $T_{ij}H$ satisfy the equilibrium equations with prescribed body forces $F_{iH} \equiv F_iH$, hence

$$T_{ij,j} + F_i = 0. \tag{45}$$

n_i are the components of a unit vector normal to A. It is not required that the stresses σ_{ij} satisfy compatibility or the boundary conditions on displacement.

The Laplace transform of the appropriate complementary functional is [1]

$$I'_\sigma = \frac{1}{2}\int_V A_{ij}^{kl}\sigma'_{ij}\sigma'_{kl}\,dV - \int_{A_u} U'_{iH}\sigma'_{ij}n_j\,dA, \tag{46}$$

where U'_{iH} is the transformed, prescribed surface displacement

$$U'_{iH} = U_i(x)/p, \tag{47}$$

and the operational compliance matrix, A_{ij}^{kl}, as given by thermodynamics [1, 2], is

$$A_{ij}^{kl} = \sum_s \frac{C_{ij}^{kl(s)}}{1 + \tau_s p} + \frac{C_{ij}^{kl}}{p} + C_{ij}''^{kl}, \tag{48}$$

and each matrix composing A_{ij}^{kl} satisfies the same properties as those composing Z_{ij}^{kl}.

The generalized stresses are obtained in the same way as the generalized coordinates in the previous discussion. Namely, the transformed stresses

$$\sigma'_{ij} = f_{ij}^\alpha Q'_\alpha + T_{ij}/p \tag{49}$$

are substituted into I'_σ and then the condition for stationary behavior, $\delta I'_\sigma = 0$, provides us with a set of N algebraic equations for the Q'_α,

$$B_{\alpha\beta}Q'_\alpha = \langle q_\beta \rangle / p, \tag{50}$$

where we define

$$B_{\alpha\beta} \equiv \int_V \left[\sum_s \frac{C_{ij}^{kl\,(s)}}{1 + \tau_s p} + \frac{C_{ij}^{kl}}{p} + C_{ij}''^{kl} \right] f_{ij}^\alpha f_{kl}^\beta\,dV \tag{51}$$

$$\langle q_\beta \rangle \equiv \int_{A_u} U_i f_{ij}^\beta n_j\,dA - \int_V \left[\sum_s \frac{C_{ij}^{kl(s)}}{1 + \tau_s p} + \frac{C_{ij}^{kl}}{p} + C_{ij}''^{kl} \right] f_{ij}^\beta T_{kl}\,dV \tag{52}$$

The similarity between the present set (50) and the previous equations (13) is evident. There are, however, two small differences. Specifically, observe that the right-hand side of (13) has a factor $1/p^2$, while the factor is $1/p$ in (50); this implies that Q'_α cannot have a double pole at the origin. As the second difference, it is noted from (50)–(52) that if $\int_{A_u} \neq 0$ and $C_{ij}''^{kl} \equiv 0$, then Q'_α does not vanish at $p = \infty$; this leads to an infinite (delta function, $\delta(t)$) stress at $t = 0$. Thus, on the basis of these remarks and in analogy with the previous Theorems I and II we can state two companion theorems:

THEOREM III. *The singularities of Q'_α occur only on the nonposititve real p-axis.*

THEOREM IV. *When the retardation times, τ_s, are constant the singularities of Q'_α are simple poles except at $-1/\tau_s$ where double poles may occur.*

When the $\langle q_\beta \rangle$ are independent of p, double poles do not occur and the time dependence of the generalized stresses is given by

$$Q_\alpha = \sum_r T_\alpha^{(r)} e^{-t/\lambda_r} + T_\alpha + T''_\alpha \delta(t) \tag{53}$$

where $T_\alpha^{(r)}$, T_α, and T''_α are constants, the vanishing of which depends on the matrices in (51) and (52), and the λ_r are positive time-constants. Since a double pole at the origin does not occur there is no term proportional to time.

4. Conclusions. It has been shown that when prescribed loads and displacements are step functions of time, use of the Ritz method of approximate analysis, in conjunction with operational analogs of the familiar potential and complementary energy principles of elasticity, leads to time-dependent displacements and stresses that are given in most cases by series of monotonically decaying exponentials. This result is consistent with the transient behavior, which was deduced by Biot [3] from different considerations, for generalized coordinates that define the motion of a linear thermodynamic system subjected to step-function forces. Also, it should be clear that the same results as derived here by the Laplace transform would have been obtained by using the convolution form of variational principles discussed in the introduction.

While our results concerning time dependence are based on the Ritz method, it seems reasonable to expect that exact analysis as well as other methods of approximation (e.g., the method of Kantorovich [7]) would provide the same characteristic exponential behavior. This appears to be quite difficult to prove, in general, but it is in agreement with numerical examples [1].

As a final point, it is recognized that in many viscoelastic problems exact inversion of transformed variables is very involved, even when Theorems I–IV apply. However, an easily applied, approximate method of inversion given in an earlier paper [8] can be used, and it is especially suited for inverting transforms which have singularities on only the real axis.

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