DISTRIBUTIONS INVOLVING NORMS OF CORRELATED GAUSSIAN VECTORS*

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Abstract. The norm of a Gaussian vector is called a Rayleigh random variable. We compute various first and higher order probability density functions of products and quotients of correlated Rayleigh variates. Moments of these distributions are also calculated. The main results are summarized in Section 2 below. Extensive use is made of formulas involving special functions. These identities enable us to obtain the desired results directly and efficiently.

1. Introduction. Various authors [1, 2, 3, 4, 5, 6] have examined properties of Rayleigh variates and considered some of their applications; in particular, see also the bibliographies in [1] and [2]. Our present objective is to extend some of the above referenced results by computing density functions of products and quotients of Rayleigh random variables.

We recall the definition of a Rayleigh random variable. Let $X = \{x_1, x_2, \cdots, x_n\}$ be an n-dimensional Gaussian random vector; that is, X has a joint n-dimensional Gaussian distribution. Then $|X|$, the norm of X, is termed a Rayleigh variate; and $|X|^2$ is essentially a chi-square random variable. If X has an arbitrary positive definite covariance matrix, then it is of course possible to write down the distribution of $|X|$ as a multiple integral. However, if we assume that X has mean vector A and diagonal covariance matrix $\psi_0 I$ (where ψ_0 is a positive constant and I is the identity matrix) then the frequency function of $|X|$ may be determined explicitly; namely,

$$g(r) = (a/\psi_0)(r/a)^{n/2}e^{-(r^2+a^2)/2\psi_0}I_{(n-2)/2}(ra/\psi_0), \tag{1.1}$$

where $r = |X|$, $a = |A|$, and I_ν is the modified Bessel function of the first kind and order ν (see, for example, [1]).

Joint Rayleigh distributions are more difficult to compute. However, we can write down an explicit density function under the following assumptions: Let $X_1, X_2, \cdots, X_p$ be n-dimensional Gaussian vectors with means zero. Let Y_j, $1 \leqq j \leqq n$, the vector composed of the jth components of the X_k, have a joint p-dimensional Gaussian distribution with positive definite covariance matrix M independent of j. Let the Y_j, $1 \leqq j \leqq n$, be independent. Then if $W = M^{-1} = (w_{kk'})_{1 \leqq k, k' \leqq p}$ has the property that $w_{kk'} = 0$ for $|k - k'| > 1$, the joint density function of $r_1, r_2, \cdots, r_p$ where $r_k = |X_k|$, $1 \leqq k \leqq p$, is

$$g(r_1, r_2, \cdots, r_p) = \frac{|W|^{n/2}}{2^{(n-2)/2}\Gamma(n/2)} r_1^{(n-2)/2} r_p^{n/2} \exp(-w_{pp}r_p^2/2)$$
$$\times \prod_{k=1}^{p-1} [|w_{k,k+1}|^{-(n-2)/2} r_k \exp(-w_{kk}r_k^2/2) I_{(n-2)/2}(|w_{k,k+1}| r_k r_{k+1})]. \tag{1.2}$$

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In particular, if $p = 2$, $r_1 = |X_1|$, $r_2 = |X_2|$,

$$g(r_1, r_2) = \frac{|W|^{n/2}(r_1 r_2)^{n/2}}{(2|w_{12}|)^{(n-2)/2}\Gamma(n/2)} \exp\left(-[w_{11}r_1^2 + w_{22}r_2^2]/2\right) I_{(n-2)/2}(|w_{12}| r_1 r_2), \tag{1.3}$$

and the condition $w_{kk'} = 0$ for $|k - k'| > 1$ becomes vacuous. For a derivation and discussion of the above formulas see [6] and [1].

2. Summary of results. (*i*) Let $u = |X_1| |X_2|$ where X_1 and X_2 are n-dimensional correlated Gaussian vectors, and where the density function of $r_1 = |X_1|$ and $r_2 = |X_2|$ is given by (1.3). Then the frequency function $h(u)$ of u is given by

$$h(u) = \frac{|W|^{n/2} u^{n/2}}{(2|w_{12}|)^{(n-2)/2}\Gamma(n/2)} I_{(n-2)/2}(u|w_{12}|) K_0(u w_{11}^{1/2} w_{22}^{1/2}), \tag{2.1}$$

where K_ν is the modified Bessel function of the second kind and order ν. See, also, [2; page 34] for a similar formula.

(*ii*) Let $\nu = |X| |Y|$ where X is n-dimensional Gaussian with mean vector A and diagonal covariance matrix $\psi_0 I$; and Y is m-dimensional Gaussian with mean vector B and diagonal covariance matrix $\psi_0' I$. Let X and Y be independent. Then the density function $\eta(\nu)$ of ν is given by

$$\eta(\nu) = \frac{4}{\nu}\left(\frac{\nu^2}{4\psi_0\psi_0'}\right)^{(n+m)/4} \exp\left[-\frac{1}{2}\left(\frac{a^2}{\psi_0} + \frac{b^2}{\psi_0'}\right)\right] \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{k!\, j!\, \Gamma(k + n/2)\Gamma(j + m/2)}$$
$$\cdot \left(\frac{\nu a^2}{4\psi_0^2}\right)^k \left(\frac{\nu b^2}{4\psi_0'^2}\right)^j \left(\frac{\psi_0}{\psi_0'}\right)^{(k-j)/2} K_{k-j+(n-m)/2}\left(\frac{\nu}{[\psi_0\psi_0']^{1/2}}\right), \tag{2.2}$$

where $a = |A|$ and $b = |B|$. This result can be found in [5]. (There seems to be an error in Eq. (11) of this reference.)

(*iii*) The moments of u are given by

$$\mathcal{E}u^p = \frac{|W|^{n/2} 2^p}{(w_{11}w_{22})^{(n+p)/2}} \frac{\Gamma^2([n+p]/2)}{\Gamma^2(n/2)} {}_2F_1\left(\frac{n+p}{2}, \frac{n+p}{2}, \frac{n}{2}; \frac{w_{12}^2}{w_{11}w_{22}}\right), \tag{2.3}$$

where ${}_2F_1(a, b, c; z)$ is the hypergeometric function. The moments of ν are given by

$$\mathcal{E}\nu^p = (4\psi_0\psi_0')^{p/2} \frac{\Gamma([n+p]/2)\Gamma([m+p]/2)}{\Gamma(n/2)\Gamma(m/2)} \exp\left[-\frac{1}{2}\left(\frac{a^2}{\psi_0} + \frac{b^2}{\psi_0'}\right)\right]$$
$$\times {}_1F_1\left(\frac{n+p}{2}, \frac{n}{2}; \frac{a^2}{2\psi_0}\right) {}_1F_1\left(\frac{m+p}{2}, \frac{m}{2}; \frac{b^2}{2\psi_0'}\right), \tag{2.4}$$

where ${}_1F_1(a, b; z)$ is the confluent hypergeometric function. The results embodied in (2.1), (2.2), (2.3), (2.4) are established in Section 3.

(*iv*) Let $v = |X_1|/|X_2|$ where X_1 and X_2 are the same correlated Gaussian vectors appearing in (*i*). Then the frequency function $p(v)$ of v is given by

$$p(v) = \frac{2|W|^{n/2} v^{n-1}(w_{11}v^2 + w_{22})}{B(n/2, n/2)[(w_{11}v^2 + w_{22})^2 - (2vw_{12})^2]^{(n+1)/2}}, \tag{2.5}$$

where $B(\mu, \nu)$ is the Beta function. See, also, [2; page 34], [7, 8], [9; page 32] for similar formulas.

(*v*) Let $\omega = |Y|/|X|$ where X and Y are the same vectors appearing in (*ii*) *but we* *assume they are both of dimension* $2n$ (instead of n and m respectively). Then the density function $\rho(\omega)$ of ω is given by

$$\rho(\omega) = (-1)^n \frac{ab}{\psi_0\psi_0'}\left(\frac{\omega}{ab}\right)^n \exp\left[-\frac{1}{2}\left(\frac{a^2}{\psi_0}+\frac{b^2}{\psi_0'}\right)\right] \times \frac{d^n}{dc^n}\left\{\frac{1}{2c}\exp\left[\frac{1}{4c}(a^2/\psi_0^2+\omega^2b^2/\psi_0'^2)\right]I_{n-1}\left(\frac{\omega ab}{2c\psi_0\psi_0'}\right)\right\}, \qquad (2.6)$$

where the above derivative is to be evaluated at $c = (\psi_0^{-1} + \omega^2\psi_0'^{-1})/2$. If $n = 1$, then ω is the ratio of two Rice variates and

$$\rho(\omega) = \frac{2\omega\psi_0\psi_0'}{\sigma^4}\exp(-[a^2\omega^2+b^2]/[2\sigma^2]) \cdot\left[\left(1+\frac{a^2\psi_0'^2+\omega^2b^2\psi_0^2}{2\psi_0\psi_0'\sigma^2}\right)I_0(\omega ab/\sigma^2)+(\omega ab/\sigma^2)I_1(\omega ab/\sigma^2)\right] \qquad (2.7)$$

where $\sigma^2 = \psi_0' + \omega^2\psi_0$.

(*vi*) The moments of v are known:

$$\mathcal{E}v^q = |W|^{n/2}\frac{w_{22}^{(q-n)/2}}{w_{11}^{(q+n)/2}}\frac{\Gamma([n+q]/2)\Gamma([n-q]/2)}{\Gamma(n/2)\Gamma(n/2)}\,{}_2F_1\left(\frac{n+q}{2},\frac{n-q}{2},\frac{n}{2};\frac{w_{12}^2}{w_{11}w_{22}}\right) \qquad (2.8)$$

(for $q < n$), see [8]. The first moment of ω [Eq. (2.7)] is given by

$$\mathcal{E}\omega = \frac{\pi}{2}\left(\frac{\psi_0'}{\psi_0}\right)^{1/2}\exp\left[-\frac{1}{2}\left(\frac{a^2}{\psi_0}+\frac{b^2}{\psi_0'}\right)\right]{}_1F_1\left(\frac{1}{2},1;\frac{a^2}{2\psi_0}\right){}_1F_1\left(\frac{3}{2},1;\frac{b^2}{2\psi_0'}\right). \qquad (2.9)$$

The results of (2.5), (2.6), (2.7) and (2.9) are derived and discussed in Section 4.

(*vii*) Let $u_1 = r_1s_1$, $u_2 = r_2s_2$ where the joint density function of r_1, r_2, s_1, s_2 is given by $g(r_1, r_2)g(s_1, s_2)$ and g is defined in (1.3). Then the joint frequency function $h(u_1, u_2)$ of u_1 and u_2 is given by

$$h(u_1, u_2) = \frac{|W|^n(u_1u_2)^{n-1}}{2^{2n-4}\Gamma^2(n/2)}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{k!\,j!\,\Gamma(k+n/2)\Gamma(j+n/2)}\left(\frac{u_1u_2w_{12}^2}{4}\right)^{k+j} \cdot K_{k-j}(u_1w_{11})K_{k-j}(u_2w_{22}). \qquad (2.10)$$

(*viii*) Let $v_1 = s_1/r_1$, $v_2 = s_2/r_2$ where r_1, r_2, s_1, s_2 are the same variates appearing in (*vii*). Then the joint frequency function $p(v_1, v_2)$ of v_1 and v_2 is given (for the case $n = 2$) by

$$p(v_1, v_2) = \frac{4|W|^2v_1v_2}{(\gamma^2-b^2)^{5/2}}[\gamma^3+2\gamma b^2+ab^2+2a\gamma^2], \qquad (2.11)$$

where

$$a = w_{12}^2(1+v_1^2v_2^2), \qquad b = 2w_{12}^2v_1v_2,$$
$$c = \tfrac{1}{2}w_{11}(1+v_1^2), \qquad \gamma = w_{11}w_{22}(1+v_1^2)(1+v_2^2)-a. \qquad (2.12)$$

(*ix*) The moments of $g(r_1, r_2)$ [*cf.* (1.3)] are known:

$$\mathcal{E}r_1^pr_2^q = \frac{2^{(p+q)/2}|W|^{n/2}}{w_{11}^{(n+p)/2}w_{22}^{(n+q)/2}}\frac{\Gamma([n+p]/2)\Gamma([n+q]/2)}{\Gamma(n/2)\Gamma(n/2)}\,{}_2F_1\left(\frac{n+p}{2},\frac{n+q}{2},\frac{n}{2};\frac{w_{12}^2}{w_{11}w_{22}}\right). \qquad (2.13)$$

(See [8] and [10].) The moments of $h(u_1, u_2)$, of course, are given by

$$\mathcal{E}u_1^p u_2^q = (\mathcal{E}r_1^p r_2^q)^2. \tag{2.14}$$

The results of (2.10) and (2.11) are treated in Section 5.

Some additional miscellaneous results appear in the later sections of this paper. In particular, a $(p - 1)$-dimensional density function of products is calculated in Section 6.

3. Uni-variate density functions of products. As our first example we shall compute the frequency function of

$$u = |X_1| \, |X_2|, \tag{3.1}$$

where X_1 and X_2 are n-dimensional correlated Gaussian vectors, and where the density function of $r_1 = |X_1|$ and $r_2 = |X_2|$ is given by (1.3). The frequency function of u is the marginal distribution

$$h(u) = \int_0^\infty \xi^{-1} g\left(\xi, \frac{u}{\xi}\right) d\xi.$$

Explicitly

$$h(u) = \frac{|W|^{n/2} u^{n/2}}{(2\,|w_{12}|)^{(n-2)/2}\Gamma(n/2)} I_{(n-2)/2}(|w_{12}|\, u) \int_0^\infty \xi^{-1} \exp\left(-[w_{11}\xi^2 + w_{22}u^2\xi^{-2}]/2\right) d\xi. \tag{3.2}$$

The identity (easily derivable from [11; page 181])

$$\int_0^\infty \xi^{2\nu-1} e^{-\alpha\xi^2} e^{-\beta\xi^{-2}} \, d\xi = \left(\frac{\beta}{\alpha}\right)^{\nu/2} K_\nu(2\alpha^{1/2}\beta^{1/2}), \qquad \alpha, \beta > 0 \tag{3.3}$$

enables us to evaluate the integral in (3.2). Thus (2.1) is established.

A somewhat similar problem has been treated in [5]. Here, in our notation, the authors consider the density function of

$$\nu = |X| \, |Y|, \tag{3.4}$$

where X is n-dimensional Gaussian with mean A and diagonal covariance matrix (equal variances), and Y is m-dimensional Gaussian with mean B and diagonal covariance matrix (equal variances). Thus (3.4) is more general than (3.1) in that the means are not assumed zero and the dimensions of X and Y are not necessarily equal. However, the authors assume that X and Y are independent. In this sense, then, their result is less general. The density function of ν is given by

$$\eta(\nu) = \int_0^\infty \xi^{-1} g(\xi) g'\left(\frac{\nu}{\xi}\right) d\xi, \tag{3.5}$$

where $g(r)$ is the density function of (1.1) and $g'(s)$ is the density function of (1.1) with n replaced by m, ψ_0 replaced by ψ_0', and a replaced by $b = |B|$. Expanding the Bessel functions, integrating term by term and invoking (3.3) we are led to (2.2).

By definition

$$\mathcal{E}u^p = \int_0^\infty u^p h(u) \, du. \tag{3.6}$$

If we expand $I_{(n-2)/2}$ in (2.1), then (3.6) may be written as

$$\mathcal{E}u^p = \frac{|W|^{n/2}}{(2\,|w_{12}|)^{(n-2)/2}\Gamma(n/2)} \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+n/2)} \left(\frac{|w_{12}|}{2}\right)^{2k+(n-2)/2} \times \int_0^\infty u^{p+n+2k-1} K_0(uw_{11}^{1/2}w_{22}^{1/2})\,du. \tag{3.7}$$

From [11; page 388],

$$\int_0^\infty x^{q-1}K_p(x)\,dx = 2^{q-2}\Gamma\left(\frac{q+p}{2}\right)\Gamma\left(\frac{q-p}{2}\right) \tag{3.8}$$

for $p^2 < q^2$. This identity enables us to evaluate the integral in (3.7) and hence we obtain (2.3).

Similarly

$$\mathcal{E}\nu^p = \int_0^\infty \nu^p \eta(\nu)\,d\nu,$$

and a direct application of (3.8) leads to (2.4). We also note that $\mathcal{E}\nu^p = \mathcal{E}\,|X|^p\,\mathcal{E}\,|Y|^p$ and hence it is not necessary to know $\eta(\nu)$ explicitly in order to calculate $\mathcal{E}\nu^p$.

4. Uni-variate density functions of quotients. We shall now compute the density function of

$$v = |X_1|/|X_2|, \tag{4.1}$$

where X_1 and X_2 are the same correlated Gaussian vectors appearing in (3.1). The density function of v is the marginal distribution

$$p(v) = \int_0^\infty \xi g(v\xi, \xi)\,d\xi, \tag{4.2}$$

where $g(r_1, r_2)$ is given by (1.3). Explicitly

$$p(v) = \frac{|W|^{n/2}\,v^{n/2}}{(2\,|w_{12}|)^{(n-2)/2}\Gamma(n/2)} \int_0^\infty \xi^{n+1} \exp\left(-\xi^2[w_{11}v^2 + w_{22}]/2\right) I_{(n-2)/2}(v\xi^2\,|w_{12}|)\,d\xi. \tag{4.3}$$

The identity [12; page 32]

$$\int_0^\infty \xi^\beta e^{-\alpha^2\xi^2} I_\nu(c\xi^2)\,d\xi = \frac{c^\nu}{2^{\nu+1}\alpha^{\beta+2\nu+1}} \frac{\Gamma([\beta+2\nu+1]/2)}{\Gamma(\nu+1)}\, {}_2F_1\left(\frac{\beta+2\nu+1}{4}, \frac{\beta+2\nu+3}{4}, \nu+1; \frac{c^2}{\alpha^4}\right) \tag{4.4}$$

is valid for $\alpha > 0$, $\alpha^2 > |c|$ and Re $(\beta + 2\nu) > -1$. Using this result in (4.3) we obtain

$$p(v) = \frac{2\,|W|^{n/2}\,v^{n-1}\Gamma(n)}{(w_{11}v^2 + w_{22})^n\Gamma^2(n/2)}\, {}_2F_1\left(\frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}; \frac{4v^2w_{12}^2}{(w_{11}v^2+w_{22})^2}\right). \tag{4.5}$$

But

$${}_2F_1\left(\frac{n}{2}, \frac{n+1}{2}, \frac{n}{2}; z\right) = {}_1F_0\left(\frac{n+1}{2}; z\right) = (1-z)^{-(n+1)/2}. \tag{4.6}$$

Thus (4.5) reduces to (2.5).

A somewhat similar problem analogous to (3.4) is to compute the density function of

$$\omega = |Y|/|X|, \tag{4.7}$$

where X is $2n$-dimensional Gaussian with mean A and diagonal covariance matrix (equal variances) and Y is $2n$-dimensional Gaussian with mean B and diagonal covariance matrix (equal variances). Thus (4.7) is more general than (4.1) in that the means are not assumed zero, but less general in the sense that we shall assume X and Y are independent and of even dimension. The density function of ω is given by

$$\rho(\omega) = \int_0^\infty \xi g'(\omega\xi)g(\xi)\, d\xi, \tag{4.8}$$

where $g(r)$ is the density function of (1.1) (with n replaced by $2n$) and $g'(r)$ is the density function of (1.1) (with n replaced by $2n$, ψ_0 replaced by ψ_0' and a replaced by $b = |B|$). Using the identity [13; page 197]

$$\int_0^\infty \xi^{2n+1}e^{-c\xi^2}I_\mu(\alpha\xi)I_\mu(\beta\xi)\, d\xi = (-1)^n \frac{d^n}{dc^n}\left[\frac{1}{2c}e^{(\alpha^2+\beta^2)/4c}I_\mu\left(\frac{\alpha\beta}{2c}\right)\right] \tag{4.9}$$

(which is valid for Re $c > 0$, Re $\mu > -1$) we arrive at (2.6) as the density function of ω.

It is clear that $p(v)$ [*cf.* (2.5)] behaves like $v^{-(n+1)}$ for v large and hence $\mathcal{E}v_q$ will exist only for $q < n$, [*cf.* (2.8)]. Similarly $\rho(\omega)$ [*cf.* (2.7)] behaves like ω^{-3} for large ω. Hence $\mathcal{E}\omega$ is the only finite integral moment. By definition

$$\begin{aligned}\mathcal{E}\omega &= \int_0^\infty \omega\rho(\omega)\, d\omega \\ &= \int_0^\infty \frac{1}{r}\left[\frac{r}{\psi_0}\exp\left(-[r^2 + a^2]/[2\psi_0]\right)I_0(ra/\psi_0)\, dr\right] \\ &\qquad \cdot \int_0^\infty s\left[\frac{s}{\psi_0'}\exp\left(-[s^2 + b^2]/[2\psi_0']\right)I_0(sb/\psi_0')\, ds\right].\end{aligned}$$

But this is a well known result (*cf.*, for example, [14; page 101]). Thus we obtain (2.9). We may also write (2.9) in terms of the modified Bessel functions of the first kind I_0 and I_1 by using the identities

$$\begin{aligned}{}_1F_1(\tfrac{1}{2}, 1; z) &= e^{z/2}I_0(z/2), \\ {}_1F_1(\tfrac{3}{2}, 1; z) = e^z\, {}_1F_1(-\tfrac{1}{2}, 1; -z) &= e^{-z/2}[(1 + z)I_0(z/2) + zI_1(z/2)],\end{aligned}$$

cf., for example, [14; pages 152 and 150].

5. Bi-variate density functions. Let us now consider the problem of finding the joint distribution of r_1s_1 and r_2s_2 where the probability density function of r_1, r_2, s_1, s_2 is given by $g(r_1, r_2)g(s_1, s_2)$ and g is defined in (1.3). If we let

$$u_1 = r_1s_1, \qquad u_2 = r_2s_2, \tag{5.1}$$

then the joint density function of u_1, u_2 may be expressed as the marginal distribution

$$\begin{aligned}h(u_1, u_2) &= \int_0^\infty \int_0^\infty \xi^{-1}\zeta^{-1}g(\xi, \zeta)g(u_1/\xi, u_2/\zeta)\, d\xi\, d\zeta \\ &= \frac{|W|^n (u_1u_2)^{n/2}}{(2\,|w_{12}|)^{n-2}\Gamma^2(n/2)}\int_0^\infty \int_0^\infty (\xi\zeta)^{-1}\exp\left(-w_{11}[\xi^2 + u_1^2\xi^{-2}]/2\right) \\ &\qquad \times \exp\left(-w_{22}[\zeta^2 + u_2^2\zeta^{-2}]/2\right)I_{(n-2)/2}(|w_{12}|\,\xi\zeta)I_{(n-2)/2}(|w_{12}|\,\xi^{-1}\zeta^{-1}u_1u_2)\, d\xi\, d\zeta.\end{aligned}$$

Expanding the Bessel functions, integrating term by term and using (3.3) we arrive at (2.10). Note the similarity of (2.10) and (2.2). This augments the observation made in

[1] that the functional form of the biased n-dimensional distribution resembles that of the unbiased $(n + 1)$-dimensional distribution.

The above results may be easily generalized to the case where the dimensions of the underlying Gaussian vectors in $g(r_1, r_2)$ and $g(s_1, s_2)$ are not necessarily equal. Also there seem to be no theoretical difficulties in generalizing (2.10) to the p-dimensional case $h(r_1s_1, r_2s_2, \cdots, r_ps_p)$ where the r's and s's each have the p-dimensional distribution of (1.2) and the r's are independent of the s's.

We turn now to the problem of finding the joint density function of s_1/r_1 and s_2/r_2 where r_1, r_2, s_1, s_2 have the same significance as in (5.1). Let

$$v_1 = s_1/r_1, \qquad v_2 = s_2/r_2. \tag{5.2}$$

Then the joint density function of v_1 and v_2 may be expressed as the marginal frequency function

$$\begin{aligned} p(v_1, v_2) &= \int_0^\infty \int_0^\infty \xi\zeta g(\xi, \zeta) g(\xi v_1, \zeta v_2)\, d\xi\, d\zeta \\ &= \frac{|W|^n (v_1v_2)^{n/2}}{(2\,|w_{12}|)^{n-2}\Gamma^2(n/2)} \int_0^\infty \int_0^\infty (\xi\zeta)^{n+1} \exp(-w_{11}\xi^2[1 + v_1^2]/2) \\ &\quad \times \exp(-w_{22}\zeta^2[1 + v_2^2]/2) I_{(n-2)/2}(|w_{12}|\,\xi\zeta) I_{(n-2)/2}(|w_{12}|\,v_1v_2\xi\zeta)\, d\xi\, d\zeta. \end{aligned} \tag{5.3}$$

The identity of (4.9) will enable us to evaluate the integral with respect to ξ in (5.3) for n even. The resulting integral with respect to ζ then poses no difficulties.

Let us explicitly compute $p(v_1, v_2)$ for the case $n = 2$. From (5.3)

$$\begin{aligned} p(v_1, v_2) = |W|^2 v_1v_2 \int_0^\infty \int_0^\infty (\xi\zeta)^3 \exp(-w_{11}\xi^2[1 + v_1^2]/2) \exp(-w_{22}\zeta^2[1 + v_2^2]/2) \\ \times I_0(|w_{12}|\,\xi\zeta) I_0(|w_{12}|\,v_1v_2\xi\zeta)\, d\xi\, d\zeta \end{aligned} \tag{5.4}$$

and from (4.9)

$$\int_0^\infty \xi^3 e^{-c\xi^2} I_0(\alpha\xi) I_0(\beta\xi)\, d\xi = \frac{1}{2c^2}\left[\left(1 + \frac{\alpha^2 + \beta^2}{4c}\right) I_0\left(\frac{\alpha\beta}{2c}\right) + \frac{\alpha\beta}{2c} I_1\left(\frac{\alpha\beta}{2c}\right)\right] e^{(\alpha^2+\beta^2)/4c}. \tag{5.5}$$

The integration with respect to ξ in (5.4) may now be performed with the aid of (5.5). There results

$$p(v_1, v_2) = \frac{|W|^2 v_1v_2}{2c^2} \int_0^\infty \left[\left(\zeta^3 + \frac{a}{4c}\zeta^5\right) I_0\left(\frac{b\zeta^2}{4c}\right) + \frac{b}{4c}\zeta^5 I_1\left(\frac{b\zeta^2}{4c}\right)\right] e^{-\gamma\zeta^2/4c}\, d\zeta, \tag{5.6}$$

where a, b, c, γ are defined by (2.12). Equation (4.4) may now be used to evaluate the integral of (5.6). We thus obtain

$$p(v_1, v_2) = \frac{4\,|W|^2 v_1v_2}{\gamma^2}\left[{}_1F_0\left(\frac{3}{2}; \frac{b^2}{\gamma^2}\right) + 3\frac{b^2}{\gamma^2}\,{}_1F_0\left(\frac{5}{2}; \frac{b^2}{\gamma^2}\right) + 2\frac{a}{\gamma}\,{}_2F_1\left(\frac{3}{2}, 2, 1; \frac{b^2}{\gamma^2}\right)\right]. \tag{5.7}$$

But

$${}_2F_1\left(\frac{3}{2}, 2, 1; z\right) = {}_1F_0\left(\frac{3}{2}; z\right) + \frac{3z}{2}\,{}_1F_0\left(\frac{5}{2}; z\right),$$

$${}_1F_0(\theta; z) = (1 - z)^{-\theta}, \qquad |z| < 1.$$

Hence (5.7) reduces to (2.11).

As a further possibility one might consider the joint distribution of v_1 and $t_2 = v_2^{-1}$ [*cf.* (5.2)]. Note that the numerator of v_1 is correlated with the denominator of t_2 and conversely. However, it is easily seen that the joint density function of v_1 and t_2 has the same form as (5.3) and hence no new results will appear.

6. Higher order distributions. Let X_k, $1 \leqq k \leqq p$, be the n-dimensional vectors described in Section 1. That is, if $r_k = |X_k|$, $1 \leqq k \leqq p$, then $g(r_1, r_2, \cdots, r_p)$ is given by (1.2). We shall compute the $(p-1)$-dimensional density function of

$$r_1 r_2,\ r_2 r_3,\ \cdots,\ r_k r_{k+1},\ \cdots,\ r_{p-1} r_p. \tag{6.1}$$

Let $u = r_1$ and

$$u_k = r_k r_{k+1}, \qquad 1 \leqq k \leqq p-1. \tag{6.2}$$

Then the joint frequency function of $u, u_1, u_2, \cdots, u_{p-1}$ is given by

$$h(u, u_1, u_2, \cdots, u_{p-1}) = \left[\prod_{k=1}^{p-1} r_k\right]^{-1} g(r_1, r_2, \cdots, r_p). \tag{6.3}$$

For convenience we shall assume first that p is odd, say $p = 2q + 1$. This restriction will be removed later. In this case (6.3) becomes

$$h(u, u_1, u_2, \cdots, u_{2q}) = \frac{|W|^{n/2} u^{n-1} \beta_q^{n/2}}{2^{(n-2)/2}\Gamma(n/2)} \exp\left[-\tfrac{1}{2}u^2 \sum_{j=0}^{q} w_{2j+1,2j+1}\beta_j^2\right]$$
$$\times \exp\left[-\tfrac{1}{2}u^{-2} \sum_{j=1}^{q} w_{2j,2j}\beta_j^{-2}u_{2j}^2\right] \prod_{k=1}^{2q} |w_{k,k+1}|^{-(n-2)/2} I_{(n-2)/2}(|w_{k,k+1}|\, u_k) \tag{6.4}$$

where we have introduced the notation

$$\beta_0 = 1, \qquad \beta_j = \prod_{i=1}^{j} u_{2i}/u_{2i-1}, \qquad 1 \leqq j \leqq q,$$

so that

$$r_{2j+1} = u\beta_j, \qquad 0 \leqq j \leqq q; \qquad r_{2j} = u^{-1}\beta_j^{-1}u_{2j}, \qquad 1 \leqq j \leqq q.$$

The distribution we desire is now the marginal density function

$$h(u_1, u_2, \cdots, u_{2q}) = \int_0^\infty h(u, u_1, u_2, \cdots, u_{2q})\, du. \tag{6.5}$$

An application of (3.3) yields

$$h(u_1, u_2, \cdots, u_{2q}) = \frac{|W|^{n/2} \beta_q^{n/2}}{2^{(n-2)/2}\Gamma(n/2)} \left[\frac{\sum_{j=1}^{q} w_{2j,2j}\beta_j^{-2}u_{2j}^2}{\sum_{j=0}^{q} w_{2j+1,2j+1}\beta_j^2}\right]^{n/4}$$
$$\times K_{n/2}\left(\left[\sum_{j=1}^{q}\sum_{i=0}^{q} w_{2j,2j}w_{2i+1,2i+1}\beta_j^{-2}\beta_i^2 u_{2j}^2\right]^{1/2}\right)$$
$$\times \prod_{k=1}^{2q} |w_{k,k+1}|^{-(n-2)/2} I_{(n-2)/2}(|w_{k,k+1}|\, u_k). \tag{6.6}$$

Suppose now p is even. Then the density function of $u_1, u_2, \cdots, u_{2q-1}$ is the marginal frequency function

$$h(u_1, u_2, \cdots, u_{2q-1}) = \int_0^\infty h(u_1, u_2, \cdots, u_{2q})\, du_{2q}. \tag{6.7}$$

But (6.6) may be written as

$$\alpha \frac{u_{2q}^{n/2}}{(u_{2q}^2 + c^2)^{n/4}} I_{(n-2)/2}(bu_{2q}) K_{n/2}(a[u_{2q}^2 + c^2]^{1/2}) \tag{6.8}$$

where a, b, c, α are independent of u_{2q}. The identity [11; page 416]

$$\int_0^\infty I_{(n-2)/2}(bt) K_{n/2}(a[t^2 + c^2]^{1/2}) \frac{t^{n/2}}{(t^2 + c^2)^{n/4}}\, dt = \frac{1}{b}\left(\frac{b}{a}\right)^{n/2} K_0(c[a^2 + b^2]^{1/2}) \tag{6.9}$$

then enables us to evaluate $h(u_1, u_2, \cdots, u_{2q-1})$. Of course, one can also start with (6.3) directly and assume p even.

An attempt to find the joint density function of

$$\frac{r_2}{r_1}, \frac{r_3}{r_2}, \cdots, \frac{r_{k+1}}{r_k}, \cdots, \frac{r_p}{r_{p-1}}, \tag{6.10}$$

where the r_i's are as in (6.1) does not seem to lead to a closed form expression.

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