4. Generalizations. The results for the wave equation can easily be generalized to include the inhomogeneous wave equation

$$c^2\nabla^2 u = \frac{\partial^2 u}{\partial t^2} + F \tag{4.1}$$

together with the mixed boundary-conditions

$$u = U \quad \text{on} \quad B_1 \times (0, \infty), \qquad \frac{\partial u}{\partial n} = V \quad \text{on} \quad B_2 \times (0, \infty) \tag{4.2}$$

and the initial conditions (2.2). Indeed, all one has to do is add the terms

$$(2/c^2 \int_R g * F * u(x, t)\, dx - 2 \int_{B_2} g * V * u(x, t)\, dx \tag{4.3}$$

to the right hand side of (2.10) and require that K be the set of all functions u which satisfy $u = U$ on $B_1 \times (0, \infty)$. The analogous assertion applies to the heat conduction problem.

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WAVE OPERATORS AND ABSOLUTELY CONTINUOUS SPECTRA*

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1. On a Hilbert space $\mathfrak{H}$ of elements $f, g, \cdots$, with inner product (f, g), let $\{E(\lambda)\}$, $-\infty < \lambda < \infty$, denote a one-dimensional spectral family. If $\mathfrak{H}_a$ denotes the Hilbert space spanned by the set of elements f for which $||E(\lambda)f||^2$ is a absolutely continuous function of λ, then $\mathfrak{H}_a$ reduces the family $\{E(\lambda)\}$ and will be called the absolutely continuous part of $\mathfrak{H}$ determined by the spectral family $\{E(\lambda)\}$; see Halmos [1], p. 104, Kato [5], p. 240, Kuroda [6], p. 436. In the case of a self-adjoint operator H or a unitary operator U, there exists in each instance a spectral family $\{E(\lambda)\}$ for which

$$H = \int_{-\infty}^{\infty} \lambda\, dE(\lambda) \quad \text{or} \quad U = \int_0^{2\pi} e^{i\lambda}\, dE(\lambda). \tag{1}$$

The restriction of H (or U) to the corresponding space $\mathfrak{H}_a$ will be called the absolutely continuous part of H (or U). The operator H or U will be called absolutely continuous on a subspace $\mathfrak{M}$ of $\mathfrak{H}$ if $\mathfrak{M} \subset \mathfrak{H}_a$.

If $\mathfrak{M}$ is a subspace its orthogonal complement in $\mathfrak{H}$ will be denoted by $\mathfrak{M}^\perp$. The space $\mathfrak{H}_s = \mathfrak{H}_a^\perp$ is spanned by the singular elements. Thus, if f is in $\mathfrak{H}_s$ and if $f \neq 0$, then the absolutely continuous part of the monotone function $||E(\lambda)f||^2$ in the Lebesgue

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decomposition is absent, that is, the total variation of $||E(\lambda)f||^2$ occurs on a set of Lebesgue measure zero.

If $\mathfrak{M}$ is a linear manifold of $\mathfrak{H}$ its closure will be denoted by $[\mathfrak{M}]$. If A is an operator its domain will be denoted by $\mathfrak{D}_A$ and its range by $\mathfrak{R}_A$.

Let H_0 and V denote self-adjoint operators and suppose that

$$H_1 = H_0 + V \tag{2}$$

is also self-adjoint. For instance, if V is bounded, its domain is $\mathfrak{H}$, and so H_1 is surely self-adjoint with domain identical to that of H_0. In the terminology of scattering theory, H_0 and H_1 correspond respectively to the free and total Hamiltonians, while V is the interaction potential; see Jauch [2], p. 134, Jauch and Zinnes [3], p. 555.

2. It will henceforth be supposed that H_0 and H_1 are unitarily equivalent, so that there exists some unitary operator U for which

$$H_1 = UH_0U^*, \quad \text{or equivalently,} \quad E_1(\lambda) = UE_0(\lambda)U^*, \tag{3}$$

where

$$H_0 = \int_{-\infty}^{\infty} \lambda \, dE_0(\lambda) \quad \text{and} \quad H_1 = \int_{-\infty}^{\infty} \lambda \, dE_1(\lambda). \tag{4}$$

It is known that under certain hypotheses on H_0 and V the strong limits

$$U_+ = \lim_{t\to\infty} U_t \quad \text{and} \quad U_- = \lim_{t\to-\infty} U_t, \quad \text{where} \quad U_t = \exp(itH_1)\exp(-itH_0), \tag{5}$$

exist, are unitary, and satisfy

$$H_1 = U_+U_0U_+^* \quad \text{and} \quad H_1 = U_-H_0U_-^*; \tag{6}$$

see [7] for references to Friedrichs, Kato, Kuroda, Rosenblum, et al.

There will be proved the following theorem, which is a generalization of a result in [8]:

(I) *Let H_0 and V be non-negative self-adjoint operators and suppose in addition that V is bounded, thus*

$$H_0 \geq 0, \qquad 0 \leq V \leq kI (k = \text{const.} > 0). \tag{7}$$

Suppose that H_1 is defined by (2) *and that* (3) *holds for some unitary operator U. Let Γ denote the smallest subspace of $\mathfrak{H}$ reducing U and containing $\mathfrak{R}_V$. Then U is absolutely continuous on Γ, that is,*

$$\Gamma \subset \mathfrak{H}_a(U). \tag{8}$$

In case V is a perturbation of rank 1, it can be supposed that

$$0 \leq V \leq kI. \tag{9}$$

It follows from (I) that if H_0 is half-bounded, if V is a one-dimensional perturbation, and if $H_0 + V = UH_0U^*$ holds for some unitary U, then U must have an absolutely continuous part. The corresponding assertion is clearly false for finite dimensional perturbations of rank different from 1, as can be seen by examples with finite matrices. Moreover, in this case, a simple trace argument shows that (3) and (9) cannot hold unless $k = 0$.

See Kato [5] who considers finite dimensional and, in some detail, one-dimensional perturbations of arbitrary (not necessarily half-bounded) self-adjoint operators.

3. Let E_n denote Euclidean n-space and let H_0 denote the unique self-adjoint extension of the symmetric operator $-\Delta$ on $L^2(E_n)$, where $\Delta u \equiv \sum_{k=1}^{n} \partial^2 u/\partial x_k^2$; see Kuroda [6], pp. 443–444 and Kato [4]. Then H_0 has the spectrum $[0, \infty)$, hence $H_0 \geq 0$, and, in addition, H_0 is absolutely continuous, that is (cf. (4)), $\mathfrak{H}_a(H_0) = \mathfrak{H}$. In case $n = 1$ so that $V = V(x)$ is a one-dimensional multiplication operator satisfying $0 \leq V(x) \leq k$ (k = const. > 0), then (7) holds and H_1 of (2) is self-adjoint with a spectrum contained in $[0, \infty)$. Under certain restrictions on $V(x)$, H_1 is also absolutely continuous with the spectrum $[0, \infty)$ and (6) holds where U_+ and U_- are unitary operators satisfying (5).

For instance, if $n = 1$, so that

$$H_0 = -d^2/dx^2 \quad \text{on} \quad \mathfrak{H} = L^2(-\infty, \infty), \tag{10}$$

this situation holds if $V(x)$ satisfies

$$0 \leq V(x) \leq k \quad \text{and} \quad \int_{-\infty}^{\infty} V(x)\, dx < \infty. \tag{11}$$

Furthermore, in this case, if $V(x)$ satisfies the second relation of (11) and also

$$0 < V(x) \leq k \tag{12}$$

almost everywhere, then the closure of $\mathfrak{R}_V$ is the entire space $\mathfrak{H}$. For this case, considered in [8], $\Gamma = \mathfrak{H}$, and so

$$\mathfrak{H}_a(U) = \mathfrak{H}, \tag{13}$$

for each of the (unitary) wave operators $U = U_+$ and $U = U_-$.

It was shown in [7] that the additional hypothesis

$$\liminf b^{-3} \int_{b_0}^{b} V^{-1}(x)\, dx = 0 \quad \text{as} \quad b \to \infty\, (b_0 \text{ fixed}) \tag{14}$$

even assures that, for any unitary U for which (3) holds, in particular for the wave operators,

$$\text{the spectrum of } U \text{ is the entire circle } |z| = 1. \tag{15}$$

It can be remarked that even for arbitrary n, if V is a radial potential, so that $V = V(r)$ with $r = (x_1^2 + \cdots + x_n^2)^{1/2}$, which satisfies (12) and (14) with $x = r$, then, whenever (3) holds for some unitary U (for instance, but not necessarily, by virtue of (6) in case U_+ and U_- exist and are unitary), necessarily (13) and (15) hold. The sufficiency of (14) for (15) in case $V = V(r)$ can be proved by methods similar to those used for the case $n = 1$.

4. In general, the relation (8) cannot be improved to

$$\Gamma = \mathfrak{H}_a(U). \tag{8'}$$

In fact, if $V = 0$ then $\Gamma = 0$ and (3) holds for any unitary U commuting with H_0 $(=H_1)$. In particular, such a unitary operator may be absolutely continuous. In the trivial case with $H_0 = I$ $(=H_1)$, relation (3) even holds for every unitary U. Another example is furnished by the absolutely continuous operator H_0 of (10); in this case $U = \exp(iH_0)$

is absolutely continuous and commutes with H_0. On the other hand, for the case $V = 0$ considered above, each of the wave operators $U = U_+$ and $U = U_-$ exists and is the identity I. Hence, in this case, (8′) does hold for $U = U_+$ and U_-, since $\Gamma = \mathfrak{H}_a(I)$ is the space consisting of the element 0 only.

If Ω denotes the smallest subspace reducing H_0 and containing $\mathfrak{R}_V$, then also Ω reduces H_1 (see Kato [5]), hence also the wave operators, and so $\Gamma \subset \Omega$. Moreover, it is clear that the restriction of U_+ and U_- to $\Omega^\perp$ is the identity operator. As a consequence of (I) there follows the theorem

(II) *If, in addition to the hypotheses of (I), the wave operators U_+ and U_- of* (5) *exist and are unitary, then $U = U_+$ or $U = U_-$ has the direct sum representation*

$$U = U_1 \oplus U_2 \oplus I \quad on \quad \mathfrak{H} = \Gamma \oplus (\Omega \ominus \Gamma) \oplus \Omega^\perp (U = U_+ \quad or \quad U_-), \tag{16}$$

where U_1 is absolutely continuous. In the special case when $\Gamma = \Omega$, relation (16) *can be refined to*

$$U = U_a \oplus I \quad on \quad \mathfrak{H} = \Omega \oplus \Omega^\perp \qquad (U = U_+ \quad or \quad U_-), \tag{16'}$$

where U_a denotes the absolutely continuous part of U.

It will remain undecided whether either or both relations (8′) and

$$\Gamma = \Omega \tag{17}$$

must always hold if $U = U_+$ or U_-. In the examples given earlier, with H_0 defined by (10) and $U = U_+$ or $U = U_-$, then $\Gamma = \Omega = 0$ when $V(x) \equiv 0$ and $\Gamma = \Omega = \mathfrak{H}$ when $V(x) > 0$. By considering direct sums of Hilbert spaces, it is easy to construct examples for which both (8′) and (17) hold for the wave operators, but where Γ is a proper subspace of $\mathfrak{H}$.

5. *Proof of* (I). The proof will depend upon a modification of the argument given in [8]. Let U have the spectral resolution

$$U = \int_0^{2\pi} e^{i\lambda}\, dE(\lambda), \tag{18}$$

and let $F(\lambda)$ be a real-valued function of period 2π with a continuous first derivative and possessing the Fourier series

$$F(\lambda) = \sum_{-\infty}^{\infty} c_k e^{ik\lambda} = c_0 + \sum_{1}^{\infty} c_k e^{ik\lambda} + \sum_{-1}^{-\infty} c_k e^{ik}. \tag{19}$$

Since $F(\lambda)$ is real, $c_{-k} = \bar{c}_k$, and so

$$F = c_0 + F^+ + \bar{F}^+, \quad \text{where} \quad F^+ = \sum_{1}^{\infty} c_k e^{ik\lambda}. \tag{20}$$

If f is arbitrary and if g is in $\mathfrak{D}_{H_0}$, then

$$\left(\int_0^{2\pi} F^+(\lambda)\, dE(\lambda) g,\ V^{1/2} f\right) = \left(\sum_{1}^{\infty} c_k V^{1/2} U^k g,\ f\right), \tag{21}$$

and hence, if $h = V^{1/2} f$ and $\sigma(\lambda) = (E(\lambda)g, h)$ (cf. [8], middle of p. 845),

$$\left|\int_0^{2\pi} F^+(\lambda)\, d\sigma(\lambda)\right| \le \|f\| \left(\sum_{1}^{\infty} |c_k|^2\right)^{1/2} (H_0 g, g)^{1/2}. \tag{22}$$

If the factors of the inner products of (21) are interchanged one obtains a relation corresponding to (22) but in which $\sigma(\lambda)$ is replaced by its conjugate. Since a complex number and its conjugate have the same absolute value, it follows that (22) remains valid if $F^+(\lambda)$ is replaced by $\bar{F}^+(\lambda)$. It then follows from (20) and (21) that

$$\left| \int_0^{2\pi} F(\lambda)\, d\sigma(\lambda) \right| \leq |c_0|\; ||V^{1/2}f||\; ||g|| + 2\; ||f|| \left(\sum_1^\infty |c_k|^2 \right)^{1/2} (H_0 g,\, g)^{1/2}. \tag{23}$$

Hence,

$$\left| \int_0^{2\pi} F(\lambda)\, d(E(\lambda)g,\, h) \right|^2 \leq C(f,\, g) \int_0^{2\pi} F^2(\lambda)\, d\lambda, \tag{24}$$

where $C(f, g)$ is a number depending on f and g but not on $F(\lambda)$. It follows that $(E(\lambda)g, h)$ is absolutely continuous whenever g is in $\mathfrak{D}_{H_0}$ and h is in $\mathfrak{R}_{V1/2}$. Since $\mathfrak{D}_{H_0}$ is dense in $\mathfrak{H}$, then $(E(\lambda)g, h)$ is absolutely continuous for all g in $\mathfrak{H}$ and h in $\mathfrak{R}_{V1/2}$ and, in particular, $||E(\lambda)h||^2$ is absolutely continuous for all h in $\mathfrak{R}_{V1/2}$. Since $[\mathfrak{R}_{V1/2}] = [\mathfrak{R}_V]$,

$$||E(\lambda)f||^2 \text{ is absolutely continuous for all } f \text{ in } [\mathfrak{R}_V]. \tag{25}$$

If $g = Uf$ then, by (18),

$$||E(\lambda)g||^2 = \int_0^\lambda d\; ||E(\mu)f||^2, \qquad 0 \leq \lambda \leq 2\pi,$$

and so $||E(\lambda)g||^2$ is absolutely continuous whenever $||E(\lambda)f||^2$ is. Similarly, the absolute continuity of $||E(\lambda)f||^2$ implies that of $||E(\lambda)g||^2$ whenever $g = U^n f$ $(n = 0, \pm 1, \pm 2, \cdots)$. Consequently, $||E(\lambda)f||^2$ is absolutely continuous for all f in the closure of the linear manifold of finite linear combinations of elements f_n in $\mathfrak{R}_{U^nV}$. Since the set of such elements f is clearly the space Γ, that is, the smallest subspace reducing U and containing $\mathfrak{R}_V$, the proof of (I) is complete.

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