THE VIBRATIONS OF A RANDOM ELASTIC STRING: THE METHOD OF INTEGRAL EQUATIONS*

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Abstract. The theory of Fredholm integral equations is applied to the problem of determining the natural frequencies of transverse vibrations of a tightly stretched elastic string whose mass per unit length varies with position in a stationary random manner. Upper and lower bounds for the statistical moments of the frequencies are given in terms of corresponding moments and appropriate correlation functions for the random linear density. The adequacy of the bounds decreases for the higher frequencies. Extensions to more general random boundary value problems are also indicated.

1. Introduction. The boundary value problem governing the free transverse vibrations of a tightly stretched elastic string having both ends fixed can be written in the form

$$U'' + \lambda(1 + A)U = 0, \qquad U(0) = U(1) = 0, \tag{1}$$

where U is the dimensionless transverse displacement, $A = A(x)$ is proportional to the random deviation of the linear density from its mean value at the point x, and λ is a parameter proportional to the square of the natural frequency. We assume that $A(x)$ is continuous, although this requirement can be relaxed, and that $1 + A > 0$. Under these circumstances the eigenvalues λ_h, ordered in increasing size, are all real and positive.

It is convenient to introduce the Green's function $T(x, y)$ for the differential operator U'' subject to the given boundary conditions. Then

$$T(x, y) = \begin{cases} x(1 - y), & 0 \le x \le y \le 1, \\ y(1 - x), & 0 \le y \le x \le 1, \end{cases} \tag{2}$$

and the boundary value problem (1) is converted into the integral equation

$$U(x) - \lambda \int_0^1 T(x, y)[1 + A(y)]U(y) \, dy = 0, \qquad 0 \le x \le 1. \tag{3}$$

With the new dependent variable $u = (1 + A)^{1/2}U$, the integral equation (3) becomes

$$u(x) - \lambda \int_0^1 K(x, y)u(y) \, dy = 0, \qquad 0 \le x \le 1. \tag{4}$$

Equation (4) is a homogeneous Fredholm integral equation of the second kind with the positive symmetric kernel

$$K(x, y) = T(x, y)[1 + A(x)]^{1/2}[1 + A(y)]^{1/2}. \tag{5}$$

The eigenvalues λ_h of Eq. (4) are the same as those of Eqs. (1), and the corresponding eigenfunctions $u_h(x)$ may be considered to be orthonormalized. If the iterated kernels $K_m(x, y)$ are defined by

$$K_1(x, y) = K(x, y),$$

$$K_m(x, y) = \int_0^1 K(x, z)K_{m-1}(z, y) \, dz, \qquad m = 2, 3, \cdots , \tag{6}$$

they admit the uniformly convergent eigenfunction expansions

$$K_m(x, y) = \sum_{h=1}^\infty \frac{u_h(x)u_h(y)}{\lambda_h^m}, \qquad m = 1, 2, \cdots . \tag{7}$$

Upon setting $y = x$ in Eq. (7) and integrating over the unit interval, we obtain the fundamental relations

$$\sum_{h=1}^\infty \lambda_h^{-m} = \int_0^1 K_m(x, x) \, dx, \qquad m = 1, 2, \cdots . \tag{8}$$

2. The determinate problem.   Equations (8) may be used in a simple way [1] to estimate the eigenvalues of the boundary value problem (1), or the integral equation (4), in the event that $A(x)$ is determinate. Thus

$$\lambda_1^{-m} = \int_0^1 K_m(x, x) \, dx - \sum_{h=2}^\infty \lambda_h^{-m}, \tag{9}$$

and an upper bound for λ_1^{-m} can be immediately obtained by dropping the infinite series on the right side of Eq. (9). This result can be improved, and lower bounds obtained as well, provided crude bounds on λ_h are already available from another source, such as a simpler version of the same problem. For example, if $|A| \le \delta < 1$, elementary comparison principles [2] yield the inequalities

$$(1 - \delta)(h\pi)^{-2} \le \lambda_h^{-1} \le (1 + \delta)(h\pi)^{-2} . \tag{10}$$

Introducing these into Eq. (9), we obtain

$$(1 - \delta)^m \sum_{h=2}^\infty (h\pi)^{-2m} \le \int_0^1 K_m(x, x) \, dx - \lambda_1^{-m} \le (1 + \delta)^m \sum_{h=2}^\infty (h\pi)^{-2m}. \tag{11}$$

The series $\sum_{h=1}^\infty (h\pi)^{-2m}$ can be summed in closed form [3] and in particular, for $m = 1, 2$,

$$\sum_{h=1}^\infty (h\pi)^{-2} = \tfrac{1}{6}, \qquad \sum_{h=1}^\infty (h\pi)^{-4} = \tfrac{1}{90}. \tag{12}$$

Thus we have the sequence of inequalities

$$(1 - \delta)(\tfrac{1}{6} - \pi^{-2}) \le \int_0^1 K(x, x) \, dx - \lambda_1^{-1} \le (1 + \delta)(\tfrac{1}{6} - \pi^{-2}), \tag{13}$$

$$(1 - \delta)^2(\tfrac{1}{90} - \pi^{-4}) \le \int_0^1 K_2(x, x) \, dx - \lambda_1^{-2} \le (1 + \delta)^2(\tfrac{1}{90} - \pi^{-4}), \cdots . \tag{14}$$

Even for $m = 1$ these bounds are markedly superior to those given by Eq. (10), and they can be improved further by using larger values of m, although the successive iterations will prove tedious for most kernels. The extension of the procedure to higher eigenvalues is obvious, at least in principle.

3. The random problem. The main result of the present section is the observation that, in the event that $A(x)$ is a random function, Eqs. (8) also yield information concerning the statistical moments of λ_1^{-1} in terms of appropriate assumptions about the statistical properties of $A(x)$. We will denote the mathematical expectation, or mean, of a random variable ξ by $\langle \xi \rangle$. Let us assume that $A(x)$ is a stationary random function, and that

$$\langle A(x) \rangle = 0, \qquad 0 \le x \le 1, \tag{15}$$

$$\langle A(x)A(y) \rangle = \mu^2 \rho(x - y), \qquad 0 \le x, y \le 1, \tag{16}$$

where, due to the stationary character of $A(x)$, $\mu^2 = \langle A^2 \rangle$ is constant, and the correlation function ρ depends only upon $x - y$, and not upon x and y separately. As a result of Eqs. (13) and (14) we have

$$\left| \left\langle \int_0^1 K(x, x)\, dx \right\rangle - (\tfrac{1}{6} - \pi^{-2}) - \langle \lambda_1^{-1} \rangle \right| \le \delta(\tfrac{1}{6} - \pi^{-2}), \tag{17}$$

$$\left| \left\langle \int_0^1 K_2(x, x)\, dx \right\rangle - (1 + \delta^2)(\tfrac{1}{90} - \pi^{-4}) - \langle \lambda_1^{-2} \rangle \right| \le 2\delta(\tfrac{1}{90} - \pi^{-4}). \tag{18}$$

Recalling that $K(x, x) = T(x, x)[1 + A(x)]$, we see that

$$\left\langle \int_0^1 K(x, x)\, dx \right\rangle = \int_0^1 T(x, x)\, dx = \tfrac{1}{6}. \tag{19}$$

Similarly,

$$K_2(x, x) = \int_0^1 T^2(x, y)[1 + A(x)][1 + A(y)]\, dy,$$

so that

$$\left\langle \int_0^1 K_2(x, x)\, dx \right\rangle = \int_0^1 \int_0^1 T^2(x, y)[1 + \langle A(x)A(y) \rangle]\, dy\, dx$$

$$= \tfrac{1}{90} + \mu^2 \int_0^1 \int_0^1 T^2(x, y)\rho(x - y)\, dy\, dx. \tag{20}$$

It is convenient to set

$$J' = \int_0^1 \int_0^1 T^2(x, y)\rho(x - y)\, dy\, dx; \tag{21}$$

Eqs. (17) and (18) then take the form

$$|\pi^{-2} - \langle \lambda_1^{-1} \rangle| \le \delta(\tfrac{1}{6} - \pi^{-2}); \tag{22}$$

$$|\mu^2 J' + \pi^{-4} - \delta^2(\tfrac{1}{90} - \pi^{-4}) - \langle \lambda_1^{-2} \rangle| \le 2\delta(\tfrac{1}{90} - \pi^{-4}). \tag{23}$$

Similar estimates of $\langle \lambda_1^{-m} \rangle$ are also available in terms of higher order correlation functions of $A(x)$. For fairly small values of δ the accuracy should be adequate for many

purposes. For instance, if $\delta = 0.1$, Eq. (22) gives the approximate value of $\pi^{-2} = 0.1013$ for $\langle \lambda_1^{-1} \rangle$ with a maximum error of 0.0065.

The most useful parameter measuring the statistical dispersion of λ_1^{-1} is its variance, defined by

$$\text{var} (\lambda_1^{-1}) = \langle \lambda_1^{-2} \rangle - \langle \lambda_1^{-1} \rangle^2. \tag{24}$$

Bounds on this quantity are also obtainable by direct substitution from Eqs. (22) and (23); thus

$$\left| \text{var} (\lambda_1^{-1}) - \mu^2 J' + \delta^2 \left(\frac{7}{180} - \frac{1}{3\pi^2} \right) \right| \leq \delta \left(\frac{1}{45} + \frac{1}{3\pi^2} - \frac{4}{\pi^4} \right). \tag{25}$$

Unfortunately, the error bound provided by Eq. (25) is unsatisfactory. In the first place, considerable precision is sacrificed by subtraction of two nearly equal quantities. Secondly, Eq. (13), from which $\langle \lambda_1^{-1} \rangle^2$ is computed, is less accurate than Eq. (14), which gives $\langle \lambda_1^{-2} \rangle$. Due to these two facts, not only may the error bound given by Eq. (25) be at least as large as the estimate of the variance, but this latter quantity may even turn out to be negative. Thus Eq. (25) is of dubious value.

In a more heuristic manner several estimates of var (λ_1^{-1}) are available when δ is small. Rewriting Eqs. (22) and (23) in the form

$$\langle \lambda_1^{-1} \rangle = \pi^{-2} + 0(\delta), \tag{26}$$

$$\langle \lambda_1^{-2} \rangle = \pi^{-4} + \mu^2 J' + 0(\delta), \tag{27}$$

we see that

$$\text{var} (\lambda_1^{-1}) = u^2 J' + 0(\delta), \tag{28}$$

a result which is also obtainable from Eq. (25). Equation (28) has one of the disadvantages alluded to before, namely that Eqs. (13) and (14), from which it is derived, do not have the same degree of precision. One way of overcoming this is to base the entire calculation upon Eq. (13), written in the form

$$\lambda_1^{-1} = \int_0^1 K(x, x) \, dx - (\tfrac{1}{6} - \pi^{-2}) + 0(\delta). \tag{29}$$

By elementary manipulations it then follows that

$$\text{var} (\lambda_1^{-1}) = \mu^2 J'' + 0(\delta), \tag{30}$$

where

$$J'' = \int_0^1 \int_0^1 T(x, x) T(y, y) \rho(x - y) \, dy \, dx. \tag{31}$$

A tentative judgment regarding the adequacy of the estimates (28) and (30) can be formed by comparing them with results obtained in other ways, such as those described in [4]. As long as $|A|$ is small, a perturbation procedure is natural. Following Collatz [5], let us consider the slightly modified problem

$$U'' + \lambda(1 + \epsilon A)U = 0, \qquad U(0) = U(1) = 0, \tag{32}$$

which reduces to (1) when $\epsilon = 1$. Assuming that the first eigenfunction and the corresponding eigenvalue have the expansions

$$U_1(x) = \sum_{h=0}^{\infty} U_{1,h}(x)\epsilon^h, \tag{33}$$

$$\lambda_1^{-1} = \sum_{h=0}^{\infty} \lambda_{1,h}\epsilon^h, \tag{34}$$

and using standard methods, we obtain

$$\lambda_{1,0} = \pi^{-2}, \tag{35}$$

$$\lambda_{1,1} = \frac{2}{\pi^2} \int_0^1 A(x) \sin^2 \pi x \, dx. \tag{36}$$

Setting $\epsilon = 1$ and keeping only the first two terms of Eq. (34), we find

$$\lambda_1^{-1} \cong \frac{1}{\pi^2}\left\{1 + 2 \int_0^1 A(x) \sin^2 \pi x \, dx\right\}. \tag{37}$$

The mean and variance of λ_1^{-1} from Eq. (37) under the previous assumptions regarding $A(x)$ are

$$\langle \lambda_1^{-1} \rangle \cong \pi^{-2}, \tag{38}$$

$$\text{var } (\lambda_1^{-1}) \cong \mu^2 \pi^{-4} J, \tag{39}$$

where

$$J = 4 \int_0^1 \int_0^1 \rho(x - y) \sin^2 \pi x \sin^2 \pi y \, dy \, dx. \tag{40}$$

Note that the estimates given by Eqs. (22) and (38) for $\langle \lambda_1^{-1} \rangle$ are identical. Those given by Eqs. (28), (30), and (39), for var (λ_1^{-1}) are of similar form; a comparison involves a choice of the correlation function $\rho(x - y)$.

4. Example. As an example of this theory, let us assume that $A(x)$ is uniformly distributed over the interval $(-\delta, \delta)$ for each x. If $f_A(t)$ is the probability density function for $A(x)$, then

$$f_A(t) = \begin{cases} \dfrac{1}{2\delta}, & |t| \le \delta, \\ 0, & |t| > \delta. \end{cases} \tag{41}$$

It follows that $\mu^2 = \delta^2/3$. Further let us assume that the correlation function $\rho(x - y)$ is given, or at least adequately approximated, by

$$\rho(x - y) = e^{-\alpha|x-y|} \tag{42}$$

where α is a nonnegative constant. This permits the evaluation of J, J', and J'' in terms of elementary integrals. The former quantity is given in [3], and J' and J'' have the form

$$J' = \frac{2}{\alpha}\left\{\frac{1}{30} - \frac{1}{6\alpha} + \frac{2}{3\alpha^2} - \frac{2}{\alpha^3} + \frac{4}{\alpha^4} - \frac{4}{\alpha^5} + \frac{4}{\alpha^5}e^{-\alpha}\right\}, \tag{43}$$

$$J'' = \frac{2}{\alpha}\left\{\frac{1}{30} - \frac{1}{3\alpha^2} + \frac{1}{\alpha^3} - \frac{4}{\alpha^5} + e^{-\alpha}\left(\frac{1}{\alpha^3} + \frac{4}{\alpha^4} + \frac{4}{\alpha^5}\right)\right\}. \tag{44}$$
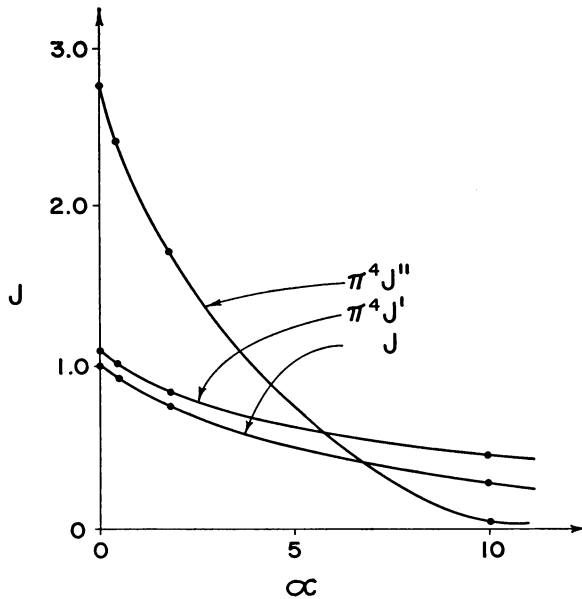
FIG. 1. The quantities J, J' and J'' plotted against α.

We have plotted, as a function of α, the quantities J, $\pi^4 J'$, and $\pi^4 J''$. The results are given in the figure.

It is clear from Fig. 1 that J' is closer to J than is J''. This tends to support the intuitive idea that use of the more accurate Eq. (14) as well as Eq. (13) is preferable to considering only the latter equation.

5. Extensions. It is clear that the method presented here can be applied to more general boundary value problems involving differential equations of the form

$$L[u] + \lambda r(x)u = 0, \tag{45}$$

where L is determinate and $r(x)$ is random, together with suitable boundary conditions. As a practical matter the Green's function must be accessible, and this tends to limit the class of operators which can be handled.

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