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A SUPERPOSITION PRINCIPLE FOR A SYSTEM OF ANALYTIC ORDINARY DIFFERENTIAL EQUATIONS*

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1. Introduction. Consider a set of differential equations of the form:

$$\dot{\eta}(t) = \sum_{i=1}^{\lambda} g_i(\eta(t), t) \quad \text{with} \quad \eta(t_0) = \eta_0 , \tag{1.1}$$

where $\eta = (\eta_1 , \cdots \eta_n)$, $g_i = (g_{i1} , \cdots , g_{in})$ and the g_i analytic. Suppose that for some of the g_i we have solutions in closed form, $\xi_i = \xi_i(t)$, to the equations $\dot{\xi}_i = g_i(\xi_i(t), t)$, $i = 1, \cdots ,\lambda$. We would like to use these "component solutions" to better approximate the solution to equation (1.1).

This paper shows that these "component solutions" permit one to write a series approximation S, to $\eta(t)$ which is a better approximation than the Taylor Series T of $\eta(t)$. That is to say, when T and S are truncated at the same order, S has a remainder which is contained as an additive term in the remainder of T. All these concepts will be made more precise below.

The principle would be especially suited for a system of ordinary differential equations arising from physical problems which are formed by a sum of simple effects. It was suggested to the author by the n-body problem of celestial mechanics. The principle might also be useful for a system of equations arising from first order partial differential equations by the method of characteristics.

2. A superposition principle.

Definition (2.1): The series $\sum_{i=0}^{\infty} (a_i + b_i)$ is said to contain the series $\sum_{i=0}^{\infty} a_i$ and the series $\sum_{i=0}^{\infty} b_i$, assuming all series under consideration are convergent.

Definition (2.2): A series S_1 is said to contain the series S_2 in a non-trivial manner if S_1 contains S_2 properly and if when S_1 contains S_2 it does not also necessarily contain $-S_2$.

Let the function $x(t)$ have two distinct series representations:

$$x(t) = \sum_{i=0}^{\infty} a_i(t),$$

$$x(t) = \sum_{i=0}^{\infty} b_i(t).$$

Let us denote the series $\sum_{i=0}^{\infty} a_i(t)$ by A and the series $\sum_{i=0}^{\infty} b_i(t)$ by B. Let a truncation of A at the $a_s(t)$ term be denoted by A_s and the remainder due to this truncation by A_s' . Similarly, denote the truncation of B at the $b_s(t)$ term by B_s , and the remainder by B_s' .

Definition (2.3): The series A is called a better approximating series to $x(t)$ than the series B if there exists an integer $\mu > 0$ such that the series A_s' is contained in a non-trivial manner in the series B_s' , for all $s \geq \mu$.

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We shall prove this principle for a special case of equation (1.1) and it will readily be seen that it also holds for the general case given in equations (1.1).

Consider the set of equations:

$$\dot{x} = f(x(t), t) + \psi(x(t), t); \qquad x(t_0) = x_0 , \tag{2.1}$$

where $x = (x_1 , \cdots , x_n)$, $f = (f_1 , \cdots , f_n)$, $\psi = (\psi_1 , \cdots , \psi_n)$ and f and ψ are analytic for $t \in I$, a spherical neighborhood about $t = t_0$.

Theorem: If the solution $\xi(t) = (\xi_1(t), \cdots , \xi_n(t))$ to the set of equations:

$$\dot{\xi}(t) = f(\xi(t), t); \qquad \xi(t_0) = x_0 \tag{2.2}$$

is known, then for $t \in I$ there exists a series representation of the solution $x(t)$ to equation (2.1) which is a better approximating series to $x(t)$ than the Taylor series of $x(t)$ about t_0.

Proof: $\xi(t)$ is clearly analytic for $t \in I$ and has a Taylor series expansion about $t = t_0$ for $t \in I$. We now show that for any integer $s \geq 1$:

$$\frac{d^s}{dt^s} x(t) = \frac{d^{s-1}}{dt^{s-1}} \psi(t) + \frac{d^{s-1}}{dt^{s-1}} f(\xi(t), t) \Big|_{\xi=x} + \sum_{\substack{\beta=0 \\ s>1}}^{s-2} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \Big|_{\xi=x} , \tag{2.3}$$

where we define

$$D^\beta(\psi, f) = \psi(x(t), t) \frac{d^\beta}{dt^\beta} \frac{\partial}{\partial \xi} f(\xi(t), t).$$[1]

Since for $s = 1$ the extreme right hand term of equation (2.3) vanishes, equation (2.3) holds for $s = 1$. Assume that it holds for some, s, $s \geq 1$, we will prove that it must then hold for $s + 1$. By induction it will then follow that equation (2.3) holds for all $s \geq 1$:

$$\begin{aligned}
\frac{d^{s+1}}{dt^{s+1}} x(t) &= \frac{d^s}{dt^s} \psi(t) + \frac{\partial}{\partial t} \left(\frac{d^{s-1}}{dt^{s-1}} f(\xi(t), t) \right)_{\xi=x} + \dot{x}(t) \frac{\partial}{\partial x} \left(\frac{d^{s-1}}{dt^{s-1}} f(\xi(t), t) \right)_{\xi=x} \\
&\quad + \sum_{\substack{\beta=0 \\ s>1}}^{s-2} \frac{d^{s-(\beta+1)}}{dt^{s-(\beta+1)}} D^\beta(\psi, f) \Big|_{\xi=x} \\
&= \frac{d^s}{dt^s} \psi(t) + \left(\frac{\partial}{\partial t} \frac{d^{s-1}}{\partial t^{s-1}} f(\xi(t), t) + [\psi(t) + \dot{\xi}(t)] \frac{\partial}{\partial \xi} \frac{d^{s-1}}{dt^{s-1}} f(\xi(t), t) \right)_{\xi=x} \\
&\quad + \sum_{\substack{\beta=0 \\ s>1}}^{s-2} \frac{d^{s-(\beta+1)}}{dt^{s-(\beta+1)}} D^\beta(\psi, f) \Big|_{\xi=x} \\
&= \frac{d^s}{dt^s} \psi(t) + \left(\frac{\partial}{\partial t} \frac{d^{s-1}}{dt^{s-1}} f(\xi(t), t) + \dot{\xi}(t) \frac{\partial}{\partial \xi} \frac{d^{s-1}}{dt^{s-1}} \ (\xi(t), t) \right)_{\xi=x} + D^{s-1}(\psi, f) \Big|_{\xi=x} \\
&\quad + \sum_{\substack{\beta=0 \\ s>1}}^{s-2} \frac{d^{s-(\beta+1)}}{dt^{s-(\beta+1)}} D^\beta(\psi, f) \Big|_{\xi=x} \\
&= \frac{d^s}{dt^s} \psi(t) + \frac{d^s}{dt^s} f(\xi(t), t) \Big|_{\xi=x} + \sum_{\substack{\beta=0 \\ s+1>1}}^{s-1} \frac{d^{s+1-(\beta+2)}}{dt^{s+1-(\beta+2)}} D^\beta(\psi, f) \Big|_{\xi=x}
\end{aligned}$$

[1]The notation $\psi(d^\beta/dt^\beta)(\partial/\partial\xi) f(\psi(t), t)$ is symbolic. If we wish to evaluate the p^{th} component of $(d^s/dt^s)x(t)$, i.e., $(d^s/dt^s)x_p(t)$, then the expression $\psi(d^\beta/dt^\beta)(\partial/\partial\xi)f(\xi(t), t)$ stands for:

$$\sum_{l=1}^{n} \psi_l \frac{d^\beta}{dt^\beta} \frac{\partial}{\partial \xi_l} f_p(\xi(t), t).$$

which is exactly equation (2.3) for $s + 1$. Since $\xi(t_0) = x_0$, we can write the Taylor series for $x(t)$ about $t = t_0$ for any $t \in I$ in the following form:

$$x(t) = x_0 + \sum_{s=1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{s-1}}{dt^{s-1}} \psi + \frac{d^{s-1}}{dt^{s-1}} f + \sum_{\substack{\beta=0 \\ s>1}}^{s-2} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \right]_{t=t_0} \tag{2.4}$$

which becomes

$$x(t) = \xi(t) + \int_{t_0}^{t} \psi(x(\eta), \eta)\, d\eta + \sum_{s=2}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-2} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \right]_{t=t_0} \tag{2.5}$$

Equation (2.4) is the Taylor series expansion of $x(t)$ which contains in a non-trivial manner the series part of equation (2.5). This means that the function series $\sum_{s=0}^{\infty} a_s(t)$ where $a_0(t) = \xi(t) + \int_{t_0}^{t} \psi(x(\eta), \eta) d\eta$ and for $s \geq 1$

$$a_s(t) = \frac{(t - t_0)^{s+1}}{(s + 1)!} \left[\sum_{\beta=0}^{s-1} \frac{d^{s-(\beta+1)}}{dt^{s-(\beta+1)}} D^\beta(\psi, f) \right]_{t=t_0} . \tag{2.6}$$

is a better approximation to $x(t)$ than the Taylor series of $x(t)$. For computing purposes, equation (2.5) does not present any significant amount of additional work, it might even conceivably reduce the amount of computing work. It turns out that the function series $\sum_{s=0}^{\infty} a_s(t)$, as defined by equation (2.6) can be so rearranged as to yield an even better approximating series to $x(t)$.[2]

Let us consider the following series in equation (2.5):

$$\rho(t) = \sum_{s=2}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-2} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \right]_{t=t_0} . \tag{2.7}$$

We will show that equation (2.7) permits the rearrangement we seek. Let us collect all terms with $\beta = s - 2$ in equation (2.7). We can then write

$$\rho(t) = \sum_{s=2}^{\infty} \frac{(t - t_0)^s}{s!} [D^{s-2}(\psi, f)]_{t=t_0} + \sum_{s=3}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-3} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \right]_{t=t_0} .$$

Suppose that we continue this rearrangement by collecting terms up to and including the $\beta = s - (m + 1)$ term. We assume that the resulting expression takes the form

$$\rho(t) = \sum_{\nu=1}^{m} \sum_{s=\nu+1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{\nu-1}}{dt^{\nu-1}} D^{s-\nu-1}(\psi, f) \right]_{t=t_0}$$
$$+ \sum_{s=m+2}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-(m+2)} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^\beta(\psi, f) \right]_{t=t_0} . \tag{2.8}$$

Since equation (2.8) holds for $m = 1$, we will prove that it holds for all $m > 1$ by induction. We now prove that equation (2.8) holds if we continue the process for $\beta =$

[2] $\sum_{s=0}^{\infty} a_s$ is still a power series and therefore may be rearranged.

$s - (m + 2)$:

$$\rho(t) = \sum_{v=1}^{m} \sum_{s=v+1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{v-1}}{dt^{v-1}} D^{s-v-1}(\psi, f) \right]_{t=t_0}$$
$$+ \sum_{s=m+2}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^m}{dt^m} D^{s-(m+2)}(\psi, f) \right]_{t=t_0}$$
$$+ \sum_{s=m+3}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-(m+3)} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^{\beta}(\psi, f) \right]_{t=t_0}$$
$$= \sum_{v=1}^{m+1} \sum_{s=v+1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{v-1}}{dt^{v-1}} D^{s-v-1}(\psi, f) \right]_{t=t_0}$$
$$+ \sum_{s=m+3}^{\infty} \frac{(t - t_0)^s}{s!} \left[\sum_{\beta=0}^{s-(m+3)} \frac{d^{s-(\beta+2)}}{dt^{s-(\beta+2)}} D^{\beta}(\psi, f) \right]_{t=t_0} .$$

By induction we now have that equation (2.8) holds for all $m \geq 1$. We can then write

$$\rho(t) = \sum_{v=1}^{\infty} \sum_{s=v+1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{v-1}}{dt^{v-1}} D^{s-v-1}(\psi, f) \right]_{t=t_0} . \tag{2.9}$$

We will now show that the inner series in equation (2.9),

$$\sum_{s=\mu+1}^{\infty} \frac{(t - t_0)^s}{s!} \left[\frac{d^{\mu-1}}{dt^{\mu-1}} \psi \frac{d^{s-\mu-1}}{dt^{s-\mu-1}} \frac{\partial}{\partial \xi} f(\xi, t) \right]_{t=t_0}$$

is the Taylor series expansion of the function

$$\gamma_\mu(\tau, t)_{\tau=t_0} = \left[\frac{d^{\mu-1}}{d\tau^{\mu-1}} \psi(x(\tau), \tau) \int_\tau^t \frac{\partial}{\partial \xi(\eta)} f(\xi(\eta), \eta)(d\eta)^{\mu+1} \right]_{\tau=t_0} , \tag{2.10}$$

where $\int_\tau^t k(\eta)(d\eta)^\mu$ denotes a μth fold integration. The function $\gamma_\mu(\tau, t)$ is a function of the two independent variables τ, t.

It is clear from equation (2.10) that $\gamma_\mu(\tau, t)$ is an analytic function of τ and t for τ, $t \in I$. As a function of t, $\gamma_\mu(\tau, t)$ has a Taylor series expansion in t, for t in some t spherical neighborhood $\hat{I}$ of any point lying on the line $t = \tau$, $\tau \in I$:[3]

$$\gamma_\mu(\tau, t) = \sum_{s=0}^{\infty} \frac{(t - \tau)^s}{s!} \frac{d^{\mu-1}}{d\tau^{\mu-1}} \psi(x(\tau), \tau) \left(\frac{d^s}{dt^s} \int_\tau^t \frac{\partial}{\partial \xi(\eta)} f(\xi(\eta), \eta)(d\eta)^{\mu+1} \right)_{t=\tau}$$

Clearly,

$$\left(\frac{d^s}{dt^s} \int_\tau^t \frac{\partial}{\partial \xi(\eta)} f(\xi(\eta), \eta)(d\eta)^{\mu+1} \right)_{t=\tau} = 0 \quad \text{for} \quad s < \mu + 1.$$

Therefore,

$$\gamma_\mu(\tau, t) = \sum_{s=\mu+1}^{\infty} \frac{(t - \tau)^s}{s!} \frac{d^{\mu-1}}{d\tau^{\mu-1}} \psi(x(\tau), \tau) \left[\frac{d^{s-\mu-1}}{dt^{s-\mu-1}} \frac{\partial}{\partial \xi} f(\xi, t) \right]_{t=\tau} .$$

[3]Strictly speaking one should use $\partial^{\mu-1}/\partial\tau^{\mu-1}$ and $\partial^s/\partial t^s$ instead of $d^{\mu-1}/d\tau^{\mu-1}$ and d^s/dt^s respectively. But because of the presence of implicit functions and the assertion that t and τ are independent variables it seems that the derivative notation, rather than the partial derivative notation is the least misleading.

In particular $\gamma_\mu(\tau, t)_{\tau=t_0}$ allows us to evaluate $\rho(t)$ as defined by equation (2.9) in the following manner:

$$\rho(t) = \sum_{\mu=1}^{\infty} \gamma_\mu(\tau, t)_{\tau=t_0} . \tag{2.11}$$

Then using equation (2.5) and (2.11) we can write $x(t)$ as

$$x(t) = \xi(t) + \int_{t_0}^{t} \psi(x(\eta), \eta)\, d\eta + \sum_{v=1}^{\infty} \gamma_v(\tau, t)_{\tau=t_0} . \tag{2.12}$$

This completes the proof of the theorem, because we can define the function series $x(t) = \sum_{v=0}^{\infty} C_v$ as follows:

$$C_0(t) = \xi(t) + \int_{t_0}^{t} \psi(x(\eta), \eta)\, d\eta, \qquad \text{and} \quad C_v(t) = \gamma_v(\tau, t)_{\tau=t_0} \quad \text{for} \quad v \geq 1.$$

This new series $\sum_{v=0}^{\infty} C_v(t)$ is a better approximating series to $x(t)$ then the series $\sum_{s=0}^{\infty} a_s(t)$ defined by equation (2.6). This is due to the way we constructed the $C_v(t)$ terms. When all terms up to and including the $\beta = s - (v + 1)$ term were collected from the series $\sum_{s=0}^{\infty} a_s(t)$, this amounted to producing the sum $\sum_{s=1}^{v} C_s(t)$. The effects on the series $\sum_{s=0}^{\infty} a_s(t)$ due to the subtraction of the $\sum_{s=0}^{v} C_s(t)$ could be summarized as follows.

The remainder due to the subtraction $\sum_{s=1}^{v} C_s(t)$ from $\sum_{s=0}^{\infty} a_s(t)$ as can be seen in equation (2.8) contains no terms $a_s(t)$ with $s \leq v$. That is, the sum $\sum_{s=0}^{v} C_s(t)$ at least as good as the truncation $\sum_{s=0}^{v} a_s(t)$ of the series $\sum_{s=0}^{\infty} a_s(t)$.

On comparing the remainder $\sum_{s=v+1}^{\infty} a_s$ with the remainder

$$\sum_{s=0}^{\infty} a_s(t) - \left[\sum_{s=0}^{v} C_s(t) \right]$$

we see that the former contains the latter for any $v \geq 1$. This is due to the fact that when $C_m(t)$ was constructed, i.e., the $\beta = s - (m + 1)$ terms were collected from

$$\sum_{s=0}^{\infty} a_s(t) - \left[\sum_{s=0}^{m-1} C_s(t) \right]$$

an additive term was subtracted from each $a_s(t)$, $s \geq m$.

Thus $\sum_{s=0}^{\infty} C_s(t)$ is a better approximating series to $x(t)$ than $\sum_{s=0}^{\infty} a_s(t)$ which in turn is a better approximating series to $x(t)$ than the Taylor series of $x(t)$.

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