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A THEORY OF NONLINEAR NETWORKS, II*

BY

R. K. BRAYTON (*International Business Machines Corporation, Yorktown Heights, N. Y.*)
AND J. K. MOSER (*Courant Institute of Mathematical Sciences, New York University*)

Introduction. In this paper we rederive the existence and the form of the mixed potential function for complete electrical networks using a graph theoretic approach. Besides some detailed proofs complementing the paper "A theory of nonlinear networks, I" [1], several additional results are obtained. Also, some well-known results for electrical networks are discussed and rederived starting with the existence of a mixed potential function. In the last section a theorem on the existence of periodic solutions for periodically excited nonlinear circuits is proved. This result can be considered as an extension of a theorem of *R.* Duffin [2].

11. Description of graphs by matrices. With any given directed graph as it was defined in section 1† one can associate a matrix (incidence matrix) in the following manner. Let the index $\mu = 1, 2, \cdots, b$ label the branches and $\nu = 1, 2, \cdots, n$ the nodes of a graph. Since every branch has an assigned direction, we can distinguish an initial node and an end node. We define $\alpha_{\nu\mu} = +1$ if the μth branch has the νth node as endpoint, $= -1$ if it is the initial point, and zero otherwise. The matrix,

$$(\alpha_{\nu\mu}), \qquad \nu = 1, \cdots, n, \qquad \mu = 1, \cdots, b, \tag{11.1}$$

which has n rows and b columns, describes the graph completely.

This matrix has several obvious properties. Since every branch connects exactly two nodes, every column of the matrix (11.1) contains exactly one pair of $+1$ and -1, and zeros otherwise. This implies, in particular,

$$\sum_{\nu=1}^{n} \alpha_{\nu\mu} = 0 \quad \text{for} \quad \mu = 1, \cdots, b, \tag{11.2}$$

so that the rank of the above matrix is at most $n - 1$.

We want to assume now that the graph is connected, i.e., that any two nodes can be connected by a path of branches. Then we will show that the rank of the matrix (11.1) is $n - 1$ (which implies also that $b \geq n - 1$).

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†Reference to sections 1–10 refer to [1].

For the proof of this statement we select a maximal tree τ in the graph. Such a tree is connected and contains all n nodes of the graph. We assume it has t branches. We select a node in τ at which only one tree branch is attached and label the node $\nu = 1$ and the branch $\mu = 1$. Clipping off this branch, we are left with a tree with $t - 1$ branches. Again we select a node of this tree with only one tree branch and label it by $\nu = 2$ and the branch by $\mu = 2$. This procedure continues until we have a tree of one branch and two nodes since $n = t + 1$.

This labeling of the tree nodes and branches has the effect that

$$\alpha_{\nu\mu} = 0 \quad \text{for} \quad \nu < \mu < n, \tag{11.3}$$

Since a branch $\mu < n$ is attached only to a node with $\nu \geq \mu$. Moreover,

$$\alpha_{\nu\nu} = \pm 1 \quad \text{for} \quad \nu = 1, \cdots, t,$$

which shows that the square matrix $(\alpha_{\nu\mu})$ with $\nu, \mu = 1, \cdots, t$ has determinant ± 1 and therefore the rank of the matrix (11.1) is equal to $t = n - 1$.

Since the rank of the matrix (11.1) is $t = n - 1$, it suffices to consider only $t = n - 1$ of its rows. The missing row can be recovered with the relations (11.2). Therefore, we cancel, for instance, the last row and introduce the matrix

$$A = (\alpha_{\nu\mu}), \qquad \nu = 1, \cdots, t, \qquad \mu = 1, \cdots, b, \tag{11.4}$$

which also describes the graph completely.

The matrix A contains in each column one pair $+1$, -1 or exactly one nonzero element. Moreover, A has maximal rank, namely $t = n - 1$. If the graph is not connected, then the rank t is not $n - 1$ any longer, but n minus the number of unconnected components, and, again, one can introduce a matrix A with maximal rank t by omitting one node from each component.

12. Kirchhoff's laws. With the matrix $(\alpha_{\nu\mu})$ of (11.1), Kirchhoff's node law takes the form

$$\sum_{\mu=1}^{b} \alpha_{\nu\mu} i_\mu = 0, \qquad \nu = 1, \cdots, n.$$

Since, on account of (11.2) one of these equations follows from the others, we can drop one. Introducing the vector $i = (i_1, \cdots, i_b)$, we have

$$Ai = 0, \tag{12.1}$$

where A is the matrix (11.4). We denote the linear space of vectors i satisfying (12.1) by $\mathcal{J}$. Moreover, if we denote the row vectors of A by

$$a_\nu = (\alpha_{\nu 1}, \cdots, \alpha_{\nu b}), \qquad \nu = 1, \cdots, t,$$

then (12.1) can be written in the form

$$(a_\nu, i) = 0, \qquad \nu = 1, \cdots, t,$$

for all vectors i in $\mathcal{J}$, i.e., the vectors $a_1, \cdots, a_t$ span the orthogonal complement of $\mathcal{J}$. We have shown in Section 2 that $\mathcal{V}$ is orthogonal to $\mathcal{J}$ and has dimension $t = n - 1$ (since we can choose t independent voltages). Hence, this shows that $\mathcal{V}$ *is the entire orthogonal complement of* $\mathcal{J}$ and that $a_1, \cdots, a_t$ form a basis for $\mathcal{V}$.

Theorem 9. The space $\mathcal{V}$ of voltage vectors satisfying Kirchhoff's voltage law is spanned by t vectors $a_1, \cdots, a_t$ with components ± 1, 0 which are the rows of the matrix (11.4).

In a similar way there is a basis

$$b_\lambda = (\beta_{\lambda 1}, \cdots, \beta_{\lambda b}), \qquad \lambda = 1, \cdots, l = b - t,$$

of l vectors in $\mathcal{J}$. We construct a basis in such a manner that all $\beta_{\lambda\mu}$ are ± 1, 0. Since $\mathcal{J}$, $\mathcal{V}$ are orthogonal complements, we have to determine l vectors b_λ so that

$$(a_\nu, b_\lambda) = 0, \qquad \nu = 1, \cdots, t, \qquad \lambda = 1, \cdots, l. \tag{12.2}$$

This system of linear equations is conveniently written in matrix notation if we combine the $a_1, \cdots, a_t$ again into the matrix

$$A = (A_1, A_2),$$

where A_1 contains the first t columns (i.e., A_1 is a $t \times t$ matrix) and A_2 contains the last $b - t = l$ columns. Similarly, we introduce

$$B = (\beta_{\lambda\mu}) = (B_1, B_2),$$

where B_2 is an $l \times l$ matrix. The relations (12.2) take the form

$$A_1 B_1^T + A_2 B_2^T = 0,$$

where B_1^T denotes the transposed matrix of B_1. Choosing $B_2 = I$, we solve this system in the form

$$B_1^T = -A_1^{-1} A_2,$$

or

$$B = (-A_2^T (A_1^T)^{-1}, I). \tag{12.3}$$

This is always possible if $\det A_1 \neq 0$, but this is just what was achieved in the last section (see 11.3).

From the properties of A one verifies that the matrix $B = (\beta_{\lambda\mu})$ so constructed has elements ± 1, 0 only.*

In what follows, it will be convenient to replace the matrix A by the matrix

$$A^* = A_1^{-1} A = (I, A_1^{-1} A_2) = (I, A_2^*).$$

Since $A_2^* = -B_1^T$, then A_2^* has elements ± 1, 0 only. In fact, A^* can be obtained from A by adding rows of A and multiplying by $+1$ or -1. From the fact that $A^* i = 0$, we read off that each of the tree branch currents

$$i_\nu = -\sum_{\mu = t+1}^{b} \alpha_{\nu\mu} i_\mu$$

can be expressed solely as linear combinations of the link branch currents with coefficients ± 1, 0. The matrices A and A^* are called cut-set matrices by Guillemin (see [4]),

*A matrix is said to be unimodular if every minor determinant equals 0, $+1$, or -1. In a theorem by Heller and Tompkins (see Hoffman and Kruskal [3]) it is shown that matrices with the properties of A are unimodular. Clearly, a nonsingular square matrix such as A_1 must have an inverse with entries 0, $+1$, or -1 only. Furthermore, the inverse has the property that the nonzero elements in each row have the same sign. From this it follows that $A_1^{-1} A_2$ has entries 0, $+1$, or -1 only.

and they differ only in the choice of an independent set of node-pair voltage variables.

The elements of B have a simple interpretation also. Namely, Kirchhoff's voltage law can be written in the form

$$Bv = 0,$$

or in components (using 12.3)

$$v_{\lambda+t} = \sum_{\mu=1}^{t} -\beta_{\lambda\mu} v_\mu , \qquad \lambda = 1, \cdots , l. \tag{12.4}$$

This relation expresses the link voltage (in the branch $\lambda + t$) in terms of the tree voltages $v_1 , \cdots , v_t$. In other words, each row of the matrix B in (12.3) corresponds to the loop through the link with label $\lambda + t$. The matrix B is called the tie-set matrix by Guillemin where the choice of the independent current variables is simply the link branch currents of the links of τ.

Summarizing, we have found in the row vectors of A^* and B basis vectors for the mutually orthogonal spaces $\mathcal{V}$ and $\mathcal{J}$ respectively. The $n \times n$ matrix

$$C = \begin{bmatrix} A^* \\ B \end{bmatrix} = \begin{bmatrix} I & A_2^* \\ -A_2^{*T} & I \end{bmatrix} \tag{12.5}$$

will be called the *connection* matrix. It is a nonsingular matrix which will be of importance in the next section.

13. Construction of the matrix γ and the mixed potential.

A. *The form of the connection matrix C for a complete circuit.* We want to show how to construct the mixed potential for a complete circuit and in the process to locate the matrix γ of section 6 as a submatrix of the connection matrix (12.5). In constructing the mixed potential, we proceed directly from the connection matrix. Recalling the situation of section 6, the graph $\mathfrak{N}$ was broken up into two graphs $\mathfrak{N}_i$ and $\mathfrak{N}_v$, $\mathfrak{N}_v$ was obtained by choosing a subtree τ' of a maximal tree τ and adding to this each link which formed a loop only through branches of τ' ($\mathfrak{N}_v = \tau' + \mathfrak{L}'$). $\mathfrak{N}_i$ was taken as the remaining $b' - t' - l'$ branches. ($\mathfrak{N}_i = \tau - \tau' + \mathfrak{L} - \mathfrak{L}'$).

We label the branches as in section 12, i.e.,

$$\begin{aligned} &\mu \in \tau', && \mu = 1, \cdots , t', \\ &\mu \in \tau - \tau', && \mu = t' + 1, \cdots , t, \\ &\mu \in \mathfrak{L} - \mathfrak{L}', && \mu = t + 1, \cdots , t + l - l', \\ &\mu \in \mathfrak{L}', && \mu = t + l - l', \cdots , t + l = b. \end{aligned}$$

We now prescribe the currents, i_σ , with $\sigma \in \mathfrak{L} - \mathfrak{L}'$, and the voltages, v_μ , with $\mu \in \tau'$ which form a complete set of variables. Recall that in a complete circuit the branches of τ' contain capacitors only, the branches of $\mathfrak{L} - \mathfrak{L}'$ inductors only, and the remaining branches resistors only. For notation let $v^{(1)}$, $v^{(2)}$, $v^{(3)}$, $v^{(4)}$, $i^{(1)}$, $i^{(2)}$, $i^{(3)}$, $i^{(4)}$ denote column vectors where the superscripts 1, 2, 3, and 4 refer to the branches of τ', $\tau - \tau'$, $\mathfrak{L} - \mathfrak{L}'$, and $\mathfrak{L}'$ respectively. Thus,

$$i = \begin{bmatrix} i^{(1)} \\ i^{(2)} \\ i^{(3)} \\ i^{(4)} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v^{(1)} \\ v^{(2)} \\ v^{(3)} \\ v^{(4)} \end{bmatrix}.$$

Note that $v^{(1)} = v^*$ and $i^{(3)} = i^*$ in the notation of Section 4, and these form the complete set.

We now partition the connection matrix C into t', $t - t'$, $l - l'$, and l' rows and columns as shown below.

$$C = \begin{array}{c} \\ t' \\ t-t' \\ l-l' \\ l' \end{array} \begin{array}{c} \begin{array}{cccc} t' & t-t' & l-l' & l' \end{array} \\ \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \end{array}$$

Lemma: For a complete circuit

$$C = \begin{bmatrix} I & 0 & c_{13} & c_{14} \\ 0 & I & c_{23} & 0 \\ -c_{13}^T & -c_{23}^T & I & 0 \\ -c_{14}^T & 0 & 0 & I \end{bmatrix}. \tag{13.1}$$

Proof: From the form of A^* and B given by (12.5), it is obvious that C has the above form except for the fact that $c_{24} = -c_{42}^T = 0$. Since we know that $c_{24} = -c_{42}^T$, we only consider the matrix B, the rows of which correspond to independent loops. In particular, the last l' rows of B correspond to the loops of the links of $\mathfrak{L}'$ and, by assumption, these loops are completely contained in $\mathfrak{N}_v$. This means that the submatrix c_{42} must be zero, which proves the lemma.

B. *Construction of P.* We denote the three remaining submatrices of C as follows:

$$c_{31} = \gamma, \qquad c_{32} = \alpha, \qquad c_{41} = -\beta, \tag{13.2}$$

which gives

$$C = \begin{bmatrix} I & 0 & -\gamma^T & \beta^T \\ 0 & I & -\alpha^T & 0 \\ \gamma & \alpha & I & 0 \\ -\beta & 0 & 0 & I \end{bmatrix}.$$

From the relations $A^*i = 0$ and $Bv = 0$ (or $C\binom{i}{v} = 0$), we find

$$\begin{aligned} i^{(1)} &= \gamma^T i^* - \beta^T i^{(4)}, \\ i^{(2)} &= \alpha^T i^*, \\ v^{(3)} &= -\gamma v^* - \alpha v^{(2)}, \\ v^{(4)} &= \beta v^*, \end{aligned} \tag{13.3}$$

where the notation $i^{(3)} = i^*$, and $v^{(1)} = v^*$ was used.

Since $i^{(1)}$ is the set of currents through the branches of τ' which contain only capacitors in a complete circuit, it follows that

$$i^{(1)} = -C \frac{dv^{(1)}}{dt} = -C \frac{dv^*}{dt}. \tag{13.4}$$

Similarly, we have

$$v^{(3)} = -L \frac{di^{(3)}}{dt} = -L \frac{di^*}{dt}, \tag{13.5}$$

since $v^{(3)}$ is the set of voltages across the branches of $\mathcal{L} - \mathcal{L}'$ and these contain only inductors.

The remaining branches contain only resistors and we use the fact that the current or voltage in a resistor is a function of *its* voltage or current alone, i.e., we can write formally

$$\begin{aligned} v^{(2)} &= -f(i^{(2)}), \\ i^{(4)} &= -g(v^{(4)}). \end{aligned} \tag{13.6}$$

Combining equations (13.3) to (13.6), we express the differential equations in terms of the variables i^* and v^* only:

$$\begin{aligned} L \frac{di^*}{dt} &= \gamma v^* - \alpha f(\alpha^T i^*), \\ C \frac{dv^*}{dt} &= -\gamma^T i^* - \beta^T g(\beta v^*). \end{aligned} \tag{13.7}$$

Because of the form of (13.7), one sees easily that

$$P(i^*, v^*) = (i^*, \gamma v^*) - \int_0^{i^*} (\alpha f(\alpha^T i^*), di^*) + \int_0^{v^*} (\beta^T g(\beta v^*), dv^*), \tag{13.8}$$

or, written in another form,

$$P(i^*, v^*) = (i^*, \gamma v^*) - \int (f(i^{(2)}), di^{(2)}) + \int (g(v^{(4)}), dv^{(4)}), \tag{13.9}$$

where the integration is from $i^{(2)} = 0$ to $\alpha^T i^*$ and from $v^{(4)} = 0$ to βv^*.

The submatrix γ of the connection matrix is therefore the same matrix γ defined by (6.3) and thus we have shown, as was promised, that γ has only elements ± 1, 0, in particular $\gamma = -c_{13}^T = c_{31}$. We see also that the current potential of $\tau - \tau'$, which we have written as $-A(i^*)$, is given by

$$\begin{aligned} A(i^*) &= \int_0^{i^*} (\alpha f(\alpha^T i^*), di^*) \\ &= \sum_\mu \int f_\mu(i_\mu)\, di_\mu \,\Big|_{i^{(2)} = \alpha^T i^*}. \end{aligned} \tag{13.10}$$

and the voltage potential of $\mathcal{L}'$, written as $-B(v^*)$, is given by

$$\begin{aligned} B(v^*) &= \int_0^{v^*} (\beta^T g(\beta v^*), dv^*) \\ &= \sum_\sigma \int g_\sigma(v_\sigma)\, dv_\sigma \,\Big|_{v^{(4)} = \beta v^*}. \end{aligned} \tag{13.11}$$

Thus $i^{(2)} = \alpha^T i^*$ and $v^{(4)} = \beta v^*$ express how the currents and the voltages in the resistors are related to the independent variables. This will prove useful in section 19 in determining the behavior of $P(i^*, v^*)$ as $| i^* | + | v^* | \to \infty$.

14. Existence of equilibria. If we assume that the network contains only resistor elements—as we discussed them in section 3—a relation between the voltage and the current is introduced, for instance,

$$i_\mu = g_\mu(v_\mu), \qquad \mu = 1, \cdots, b. \tag{14.1}$$

It is then of basic importance to verify that these relations are compatible with Kirchhoff's laws and that a simultaneous solution of equations (14.1) and Kirchhoff's laws exists, i.e., an equilibrium exists.

Theorem 10: If in (14.1) we assume that g_μ is a continuous function of v_μ and

$$\int_0^{v_\mu} g_\mu(u)\, du \to \infty \quad \text{for} \quad | v_\mu | \to \infty, \qquad \mu = 1, \cdots, b,$$

then there exists a solution to equations (14.1) and Kirchhoff's laws.

Proof: The function (voltage potential)

$$G(v) = \sum_{\mu=1}^{b} \int_0^{v_\mu} g_\mu(u)\, du,$$

where $v = (v_1, \cdots, v_b)$, is a continuously differential function which tends to ∞ as $| v |$ tends to ∞ in any direction. If we restrict the vector v to the linear subspace $\mathcal{V}$ given by

$$Bv = 0,$$

(i.e., v satisfies Kirchhoff's voltage laws), then clearly G will also tend to ∞ there. The proof of this theorem rests on the fact that the extreme values of G restricted by $Bv = 0$ automatically satisfy Kirchhoff's current laws. Namely, if $v = v^0$ corresponds to an extremum of G on $\mathcal{V}$, then the gradient

$$\left. \frac{\partial G(v)}{\partial v_\mu} \right|_{v=v^0} = g_\mu(v_\mu^0) = i_\mu, \qquad \mu = 1, \cdots, b,$$

belongs to the orthogonal complement $\mathcal{J}$ of $\mathcal{V}$. To show this, we write the side conditions $Bv = 0$ in the form $(b_\nu, v) = 0$ where the b_ν are the rows of B. According to the Lagrange multiplier method, we must find the extremum of $G(v) - \sum_{\nu=1}^{n} \lambda_\nu (b_\nu, v)$, which gives the condition $G_v - \sum_{\nu=1}^{n} \lambda_\nu b_\nu = 0$. Hence, G_v lies in the space spanned by $b_1, \cdots, b_n$, i.e., in $\mathcal{J}$. In fact, this property is necessary and sufficient for an extremum of G on $\mathcal{V}$.

Therefore, the proof of the theorem is reduced to showing the existence of an extremum of the function G restricted to $\mathcal{V}$. However, since $G \to \infty$ as $| v | \to \infty$ in $\mathcal{V}$, then certainly G possesses a minimum in $\mathcal{V}$ which completes the proof.

For uniqueness one must ensure that there are no other extrema except the minimum. For this purpose, it suffices to assume that G is convex or $g_\mu' > 0$. This corresponds to "positive" resistors (also called quasi-linear resistors by Duffin [5].)

The above argument required that the i_μ were single-valued functions of v_μ. This need not be the case for nonlinear resistors. In some cases, v_μ may be a single-valued function of i_μ

$$v_\mu = f_\mu(i_\mu), \tag{14.2}$$

and then the existence of an equilibrium can be established in an analogous fashion by finding a minimum of

$$F(i) = \sum_{\mu=1}^{b} \int_0^{i_\mu} f_\mu(\lambda)\, d\lambda \tag{14.3}$$

(current potential) restricted to the linear space $\mathcal{J}$. In fact, this description can be considered dual to the first one.

We see that either of Kirchhoff's laws can be replaced by a law stating that a certain potential should be a minimum. This fact is, of course, well known. For instance, Maxwell [6] stated such a theorem in 1873 for linear networks called Maxwell's "Minimum Heat Theorem" and in 1951 W. Millar [7] proved the corresponding statement for nonlinear networks. Millar uses the terms "content" and "co-content" which in this paper are called "current" and "voltage potential", respectively. Such concepts were also used by Duffin [5] to prove existence of an equilibrium solution for nonlinear networks and the uniqueness for quasi-linear networks.

15. n-Ports and reciprocity of networks.

A. *n-Ports.* An n-port R-circuit can be defined as a network containing only resistive elements and n additional free branches in which either the currents or the voltages can be prescribed—by idealized current or voltage sources. We depict such an n-port by a box (see figure 16) containing the resistors and n pairs of free wires. For

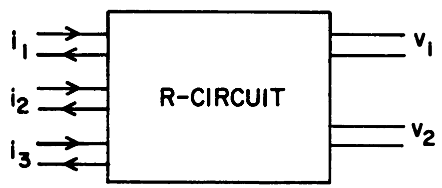

FIG. 16. n-Port R-circuit.

definiteness we insert current sources in the r pairs on the left, prescribing the currents $i_1, \cdots, i_r$ and voltage sources at the right with voltages $v_1, \cdots, v_s$.

The equilibrium state of the n-port will then depend on the parameters $i_1, \cdots, i_r$, $v_1, \cdots, v_s$. The effect of the network on the free ends is described by the functions $V_1, \cdots, V_r$ and $I_1, \cdots, I_s$ which describe the voltages and currents, respectively, at the free ends. We choose the notation in such a way that (i_ρ, V_ρ) $(\rho = 1, \cdots, r)$ correspond to the same free end and similarly for (v_σ, I_σ) $(\sigma = 1, \cdots, s)$.

We will assume that these functions are well-defined and single-valued. Then, it follows from section 4 that these functions can be obtained as derivatives of a single function $P(i, v)$—the potential of the n-port. Namely, if j_μ, w_μ $(\mu = 1, \cdots, b)$ denote the currents and voltages in the interior branches of the n-port, then we have from theorem 2 of section 2

$$\int_\Gamma \left[\sum_{\mu=1}^{b} w_\mu\, dj_\mu + \sum_{\rho=1}^{r} V_\rho\, di_\rho + \sum_{\sigma=1}^{s} v_\sigma\, dI_\sigma \right] = 0,$$

where the integration is taken over the path Γ of equilibrium solutions from a fixed point to a variable one. Therefore, with

$$P = \int_\Gamma \sum_{\mu=1}^{b} w_\mu\, dj_\mu + \sum_{\sigma=1}^{s} \int_\Gamma d(v_\sigma I_\sigma),$$

(which is to be expressed as a function of i, v), we have

$$dP + \sum_{\rho=1}^{r} V_\rho \, di_\rho - \sum_{\sigma=1}^{s} I_\sigma \, dv_\sigma = 0,$$

or

$$V_\rho = -\frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r,$$

$$I_\sigma = \frac{\partial P}{\partial v_\sigma}, \qquad \sigma = 1, \cdots, s. \tag{15.1}$$

Thus, we see that this function P can be considered as characteristic for the n-port.

How the function P can be constructed has been considered in detail in sections 5 and 6 and need not be repeated here. Obviously, the situation discussed there is obtained if one inserts inductors and capacitors in the free ends of the n-port.

B. *Reciprocity of networks.* If we consider an n-port at which only currents are prescribed, $i_1, \cdots, i_r$ ($r = n$, $s = 0$), then the voltages $V_1, \cdots, V_r$ are given by

$$V_\rho = -\frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r. \tag{15.2}$$

In case all the resistors are linear, these relations are linear, i.e., there exist some constants $R_{\rho\sigma}$ such that

$$V_\rho = \sum_{\sigma=1}^{r} R_{\rho\sigma} i_\sigma, \qquad \rho = 1, \cdots, r. \tag{15.3}$$

From the form of (15.3) it is clear that the constants $R_{\rho\sigma}$ can be found by

$$R_{\rho\sigma} = \left(\frac{V_\rho}{i_\sigma}\right)_{i_{\sigma'}=0,\ \sigma\neq\sigma'}. \tag{15.4}$$

The reciprocity theorem for linear passive bilateral networks (see, for instance, Guillemin [4]) is equivalent to the statement that

$$R_{\rho\sigma} = R_{\sigma\rho}.$$

This relation is obvious for the networks considered since

$$R_{\rho\sigma} = \frac{\partial V_\rho}{\partial i_\sigma} = -\frac{\partial^2 P}{\partial i_\rho \, \partial i_\sigma} \tag{15.5}$$

clearly displays this symmetry property. In the nonlinear case we will define $R_{\rho\sigma}$ by (15.5) which, in the linear case, agrees with the usual definition (15.4). Hence, the reciprocity theorem, as stated above, holds for nonlinear networks as well if the mutual resistance is defined by (15.5).

This property of reciprocity is, in fact, characteristic for the existence of a function P such that (15.2) holds. Namely, if $R_{\rho\sigma} = R_{\sigma\rho}$, then $\partial V_\rho/\partial i_\sigma = \partial V_\sigma/\partial i_\rho$ by definition, and this condition implies that V_ρ ($\rho = 1, \cdots, r$) can be considered as the gradient of some function $P(i)$. Thus, the integrability conditions imposed by (15.2) on the voltages V_ρ are—from the physical point of view—equivalent to the requirement of reciprocity for the circuit.

16. Legendre and y-Δ transformations. For an n-port R-circuit there will, in general, exist n relations between the $2n$ variables, $i_1, \cdots, i_n, v_1, \cdots, v_n$, such that n of them can be considered independent. Geometrically, this means that in the $2n$-dimensional space we have an n-dimensional surface $\sum$ which is, in fact, characteristic for the external electrical behavior of the n-port.

If, for two n-ports, these functional relations are the same, i.e., the corresponding surfaces coincide, then we will call these n-ports "equivalent." The reason for this definition is that two "equivalent" n-ports operating in a network cannot be distinguished (except by making some internal measurements).

There are several ways in which the independent variables for an n-port can be chosen. For instance, for a 2-port, one can prescribe the currents (i_1, i_2) in both free ends, or the voltages (v_1, v_2), or one voltage and one current (i_1, v_2), (i_2, v_1). We want to investigate how the corresponding potential functions are related if they exist.

We use a different notation from the last section and denote the currents and voltages at the ports by i_ν, v_ν $(\nu = 1, \cdots, n)$, i.e., we do not capitalize the letters corresponding to the dependent variables.

A simple case is when $i_1, \cdots, i_n$ are prescribed (current sources in the free ends) and the voltages are to be determined. If we denote the corresponding potential by $P = F(i_1, \cdots, i_n)$—in agreement with the notation of section 6—then we have

$$v_\nu = -\frac{\partial F}{\partial i_\nu}, \qquad \nu = 1, \cdots, n. \tag{16.1}$$

Similarly, if the voltages are prescribed and the corresponding voltage potential is denoted by $G(v_1, \cdots, v_n)$, we find

$$i_\nu = -\frac{\partial G}{\partial v_\nu}, \qquad \nu = 1, \cdots, n. \tag{16.2}$$

The two sets of relations (16.1) and (16.2) describe the same surface $\sum$ and they can be considered as transformations which are inverse to each other. The form of these two transformations is that of "Legendre transformations" since the right-hand sides are gradients of one function. For such transformations the relation between F and G can be easily expressed by the formula

$$F + G = -\sum_{\nu=1}^{n} i_\nu v_\nu . \tag{16.3}$$

To prove this we set

$$G = -F - \sum_{\nu=1}^{n} i_\nu v_\nu ,$$

and form the differential considering i_ν, v_ν as independent variables:

$$dG = -dF - \sum_{\nu=1}^{n} i_\nu \, dv_\nu - \sum_{\nu=1}^{n} v_\nu \, di_\nu$$

$$= \sum_{\nu=1}^{n} \left(-\frac{\partial F}{\partial i_\nu} - v_\nu\right) di_\nu - \sum_{\nu=1}^{n} i_\nu \, dv_\nu .$$

The first term vanishes if (16.1) holds so that

$$dG = -\sum_{\nu=1}^{n} i_\nu \, dv_\nu ,$$

which leads to (16.2). Thus, (16.1) and (16.2) are inverse Legendre transformations and the potentials are related by (16.3) (up to an additive constant).

As an example, in the linear case F is a quadratic form

$$F(i) = -\tfrac{1}{2} \sum_{\nu,\mu=1}^{n} a_{\nu\mu} i_\nu i_\mu = -\tfrac{1}{2}(i, Ai), \tag{16.4}$$

where A is a symmetric nonsingular matrix. Then

$$v = -\frac{\partial F}{\partial i} = Ai,$$

and

$$G(v) = -(i, v) - F(i) = -(i, Ai) + \tfrac{1}{2}(i, Ai)$$
$$= -\tfrac{1}{2}(i, Ai),$$

or with $v = Ai$

$$G(v) = -\tfrac{1}{2}(v, A^{-1}v). \tag{16.5}$$

In this connection, we want to discuss the well-known $Y - \Delta$ transformation and show that it has no analog for nonlinear circuits.

A Y-circuit can be considered as a 2-port with three resistors as shown in figure 17.

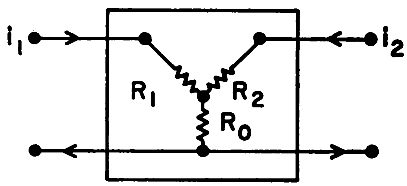

FIG. 17. Y-circuit.

We describe this circuit by the current potential

$$F = -\tfrac{1}{2} \sum_{\nu=0}^{2} R_\nu i_\nu^2 , \quad \text{where} \quad i_0 = i_1 + i_2 , \tag{16.6}$$

so that the matrix A of (16.4) has the form

$$A = \begin{bmatrix} R_1 + R_0 & R_0 \\ R_0 & R_2 + R_0 \end{bmatrix}, \tag{16.7}$$

where R_0 , R_1 , and R_2 are positive.

A Δ-circuit can also be considered as a 2-port with three conductors as shown in figure 18. We describe the circuit by the voltage potential

$$G = -\tfrac{1}{2} \sum_{\nu=0}^{2} G_\nu v_\nu^2 = -\tfrac{1}{2}(v, Bv), \tag{16.8}$$

where $v_0 = v_2 - v_1$,

$$B = \begin{bmatrix} G_1 + G_0 & -G_0 \\ -G_0 & G_2 + G_0 \end{bmatrix}, \tag{16.9}$$

and G_0 , G_1 , and G_2 are positive.

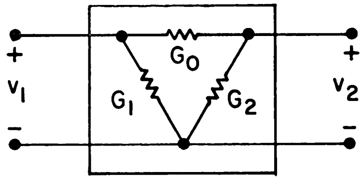

Fig. 18. Δ-circuit.

To show the "equivalence" of these two 2-ports, we must find G_0 , G_1 , $G_2 > 0$ so that G and F give rise to inverse Legendre transformations, i.e., as was shown by (16.5), that $B = A^{-1}$. The inverse of A exists since its determinant

$$R_0(R_1 + R_2) + R_1R_2$$

is positive. It remains to be shown that the elements G_ν are positive. This elementary fact follows from

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}^{-1} = \frac{1}{ac - b^2} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix}. \tag{16.10}$$

In the case under consideration $a, c > 0$ and $0 < b < \text{Min}\,(a, c)$, and one reads off that $G_0 > 0$ and $G_\nu + G_0 > G_0$ ($\nu = 1, 2$), i.e., G_0 , G_1 , $G_2 > 0$.

Similarly, for G_0 , G_1 , $G_2 > 0$, one can find the R_0 , R_1 , R_2 such that A is the inverse of B showing the "equivalence" of Y- and Δ-circuits. Using (16.10) one gets an explicit relation between the R_ν and the G_ν .

If, however, the resistors are described by nonlinear functions $w_\nu = -f_\nu(i_\nu)$ ($\nu = 0, 1, 2$) for the Y-circuit and $j_\nu = -g_\nu(v_\nu)$ ($\nu = 0, 1, 2$) for the Δ-circuit, we will show that such an equivalence does not hold any longer. We assume that $f_\nu' > 0$. Denoting the integrals of $f_\nu(\lambda)$, $g_\nu(\lambda)$ by $F_\nu(\lambda)$, $G_\nu(\lambda)$, respectively, the potential $F(i)$ of the Y circuit takes the form

$$-F(i) = F_1(i_1) + F_2(i_2) + F_0(i_1 + i_2),$$

and the potential G of the Δ-circuit takes the form

$$-G(v) = G_1(v_1) + G_2(v_2) + G_0(v_2 - v_1). \tag{16.11}$$

To show the "equivalence" of the two circuits amounts to showing that the (convex) function $F(i)$ gives rise to a function $G = -F(i) - i_1v_1 - i_2v_2$ which has the form (16.11). For a function to be in the form (16.11) it is necessary that

$$\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right) \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_2} G = 0.$$

This expression can be explicitly calculated and, in case $f_\nu = -R_\nu i_\nu$ ($\nu = 1, 2$) are linear and only $f_0(i_0)$ is nonlinear, the calculation using $G = -F(i) - i_1v_1 - i_2v_2$ gives

$$\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right) \frac{\partial}{\partial v_1} \frac{\partial}{\partial v_2} G = \frac{R_1R_2f_0''(R_1 + R_2)}{(R_1R_2 + (R_1 + R_2)f_0')^3},$$

showing that $f_0'' = 0$ necessarily. This implies that f_0 must also be linear. Thus we have shown:

A Y-circuit with two linear resistors and a third element can be equivalent to a Δ-circuit if and only if the third element is also linear.

Finally, we want to show the relation between different mixed potential functions of an n-port. Let $P(i, v)$ be the potential function for the case where $i_1, \cdots, i_r, v_{r+1}, \cdots, v_{r+s}$ $(r + s = n)$ are prescribed. Then we know from section 6 that

$$v_\rho = -\frac{\partial P}{\partial i_\rho}, \qquad \rho = 1, \cdots, r,$$

$$i_\sigma = \frac{\partial P}{\partial v_\sigma}, \qquad \sigma = r + 1, \cdots, r + s.$$

Assuming that the complementary set of variables $v_1, \cdots, v_r, i_{r+1}, \cdots, i_{r+s}$ can be prescribed, i.e., that the above transformation can be inverted, we denote the corresponding mixed potential by Q. Then

$$i_\rho = \frac{\partial Q}{\partial v_\rho}, \qquad \rho = 1, \cdots, r,$$

$$v_\sigma = -\frac{\partial Q}{\partial i_\sigma}, \qquad \sigma = r + 1, \cdots, r + s,$$

and the relation

$$P + Q = \sum_{\rho=1}^{r} i_\rho v_\rho - \sum_{\sigma=r+1}^{r+s} i_\sigma v_\sigma$$

holds (up to an additive constant). This follows from

$$\begin{aligned} dQ &= \sum_{\rho=1}^{r} Q_{v_\rho}\, dv_\rho + \sum_{\sigma=r+1}^{r+s} Q_{i_\sigma}\, di_\sigma \\ &= \sum_{\rho=1}^{r} i_\rho\, dv_\rho - \sum_{\sigma=r+1}^{r+s} v_\sigma\, di_\sigma \\ &= d\Big(\sum_{\rho\le r} i_\rho v_\rho - \sum_{\sigma>r} i_\sigma v_\sigma\Big) - dP. \end{aligned}$$

All these relations stem from the fact that

$$\sum_{\nu=1}^{n} v_\nu\, di_\nu \tag{16.12}$$

is the differential of a function. Using the notation of the calculus of differential forms, one can express this fact in the following form:

Theorem 11. If i_ν, v_ν $(\nu = 1, \cdots, n)$ are the currents and voltages of the n-port, then

$$\sum_{\nu=1}^{n} (dv_\nu \wedge di_\nu) = 0$$

(i.e., this form vanishes identically on the characteristic surface $\sum$ of the n-port introduced at the beginning of this section).

Remark. This differential form is meant in the sense of H. Cartan [8], namely: if, for example $v_\nu = v_\nu(x, y)$, $i_\nu = i_\nu(x, y)$ $(\nu = 1, 2)$ depend on two variables x, y (so as to comply with the n-port equations), then this relation means that $\sum_{\nu=1}^{2} \partial(v_\nu, i_\nu)/\partial(x, y) \equiv 0$. For instance, if $v_\nu = f_\nu(i_\nu)$ and $x = i_1$, $y = i_2$, then this yields

$$\frac{\partial(f_1, i_1)}{\partial(i_1, i_2)} + \frac{\partial(f_2, i_2)}{\partial(i_1, i_2)} = \frac{\partial f_1}{\partial i_2} - \frac{\partial f_2}{\partial i_1} = 0.$$

Proof. Denote by i_μ, v_μ ($\mu = n + 1, \cdots, n + b$) the currents and voltages of the internal branches. Then by theorem 1, we have

$$\sum_{\nu \le n} v_\nu \, di_\nu + \sum_{\mu > n} v_\mu \, di_\mu = 0.$$

Hence

$$\sum_{\nu \le n} dv_\nu \wedge di_\nu + \sum_{\mu > n} dv_\mu \wedge di_\mu = 0.$$

Since the v_μ depend on i_μ alone ($\mu > n$), it follows that $dv_\mu \wedge di_\mu = 0$ ($\mu > n$), proving the theorem.

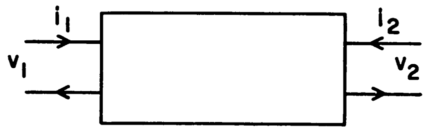

FIG. 19. 2-port network.

For example, in a 2-port as in figure 19, we have

$$di_1 \wedge dv_1 + di_2 \wedge dv_2 = 0,$$

or

$$di_1 \wedge dv_1 = d(-i_2) \wedge dv_2 ,$$

which expresses that (i_1, v_1) and $(-i_2, v_2)$ are related by an area preserving transformation. For linear circuits this transformation is written

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} v_2 \\ i_2 \end{bmatrix},$$

where as a consequence of the area preserving property $AD - BC = 1$ (see G. Newstead [9]).

17. Transformation properties of the differential equations. It must be noted that the particular form of the equations derived in section 4 depends on the choice of variables $x = \binom{i}{v}$. If one transforms the system into a new set of variables, say $y = \phi(x)$, then, in general, this form

$$J \frac{dx}{dt} = -\frac{\partial P}{\partial x}, \qquad J = \begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix}, \tag{17.1}$$

will be destroyed. Therefore, we wish to study which transformations $y = \phi(x)$ have the property that in the new coordinates the form of the equations is again

$$J \frac{dy}{dt} = -\frac{\partial Q}{\partial y}, \tag{17.2}$$

where $Q(y) = P(x)$, $y = \phi(x)$.

Theorem 12. A coordinate transformation $y = \phi(x)$ preserves the form of the differential equations (17.1) if and only if it preserves

$$(dx, J\, dx) = -(di, L\, di) + (dv, C\, dv)^*$$

*This differential is not meant in the sense of Cartan.

i.e., $(dx, J\ dx) = (dy, J\ dy)$. An equivalent condition for the Jacobian matrix ϕ_x is

$$\phi_x^T J \phi_x = J.$$

Proof. The transformed differential equations are

$$J \frac{dy}{dt} = J\phi_x \frac{dx}{dt} = -J\phi_x J^{-1} P_x .$$

On the other hand, if $P(x) = Q(y)$, we have

$$dP = (P_x , dx) = (Q_y , \phi_x\, dx),$$

or

$$P_x = \phi_x^T Q_y .$$

Our requirement is that

$$J\phi_x J^{-1} \phi_x^T = I,$$

or

$$\phi_x^T J \phi_x = J,$$

which is necessary and sufficient that (17.1) is transformed into (17.2).

Since

$$(dy, J\ dy) = (\phi_x\, dx, J\phi_x\, dx) = (dx, \phi_x^T J \phi_x\, dx),$$

the above condition is equivalent to

$$(dy, J\ dy) = (dx, J\ dx),$$

which was to be proved.*

Therefore, the form

$$(dx, J\ dx) = -(di, L\ di) + (dv, C\ dv)$$

will be associated with (17.1). This form is, in general, indefinite. However, if the system does not contain any inductors, the form reduces to

$$(dv, C\ dv) = (ds)^2,$$

which is positive definite and can be used to define a metric $(ds)^2$.

In fact, if we differentiate P along a solution, we find

$$\frac{dP}{dt} = \left(P_x , \frac{dx}{dt}\right) = -\left(\frac{dx}{dt} , J \frac{dx}{dt}\right).$$

---

*It is interesting to contrast theorem 12 with the analogous theorem for Hamiltonian systems. In such systems the differential equations have the form

$$-J^*\, dw/dt = H_w(t, w),$$

where $w = \binom{u}{v}$, $J^* = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and u, v are n-dimensional vectors. The question is asked under which conditions a transformation, $w = \phi(t, z)$ transforms a Hamiltonian system into a Hamiltonian system. The answer is that the matrix ϕ_z must satisfy the relation

$$\phi_z^T J^* \phi_z = J^*.$$

Such matrices ϕ_z are called symplectic and the corresponding transformations $w = \phi(t, z)$ are called canonical transformations.

When no inductors are present, this can be written as

$$\frac{dP}{dt} = -\left(\frac{ds}{dt}\right)^2,$$

exhibiting that P is a decreasing function of time.

The main idea of section 8 was to find a metric, i.e., a positive definite differential form, associated with the system.

As a result of theorem 12, we see that no new differential form is obtained by a transformation of coordinates which preserves the form (17.1). However, in section 8 we saw that under special assumptions a metric could be obtained by expressing the equations with a different (J, P) while keeping the coordinates the same, i.e., we found (J^*, P^*) such that

$$J^* \frac{dx}{dt} = -P^*_x,$$

where the differential

$$(dx, J^* dx)$$

was positive definite. In this case

$$\frac{dP^*}{dt} = -\left(\frac{dx}{dt}, J^* \frac{dx}{dt}\right) = -\left(\frac{ds}{dt}\right)^2,$$

which shows that P^* is a decreasing function of time.

18. Foster's reactance theorem. In the theory of linear electrical networks the study of the driving point admittance* is of basic importance. For linear circuits without resistors, Foster [10] succeeded in giving a complete description of the driving point admittances which are realized by such networks. We want to rederive his result—at least in one direction—and show that the driving point admittance has simple poles on the imaginary axis with nonnegative residues. That a circuit can be constructed for any such function is easily shown and can be found in the same paper of Foster (see also Guillemin [11]).

We consider a nonresistive circuit, i.e., we assume that $P = (i, \gamma v)$ and the differential equations take the form:

$$L \frac{di}{dt} = \gamma v + E,$$

$$C \frac{dv}{dt} = -\gamma^T i,$$

where E is a vector of the form

$$E = \begin{pmatrix} E_1(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

*The driving point admittance of a point of a network is the ratio of the resulting current at that point to an impressed voltage at the same point.

The problem is to determine $i_1(t)$ for a given function $E_1(t)$. This problem is solved easily by the use of Laplace transforms reducing it to the case of exponential functions for $E_1(t)$. Replacing all functions by exponentials, we find with $E_1 = E^* e^{pt}$, $i = i^* e^{pt}$, $v = v^* e^{pt}$ the condition

$$Mw = \begin{bmatrix} E^* \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{with} \quad w = \begin{bmatrix} i^* \\ v^* \end{bmatrix},$$

where

$$M = M(p) = \begin{bmatrix} Lp & -\gamma \\ \gamma^T & Cp \end{bmatrix}.$$

One obtains (using Cramer's rule)

$$i_1^* = \frac{M_{11}(p)}{\det M(p)} E^*,$$

where M_{11} is the subdeterminant obtained from $M(p)$ by cancelling the first row and column. In fact, the function

$$Y(p) = \frac{M_{11}(p)}{\det M(p)}$$

is the driving point admittance of the first branch.

We want to show that $Y(p)$ has poles only on the imaginary axis which are simple and have nonnegative residues. For this purpose we note that $Y(p)$ is the element in the first row and first column of the matrix $M^{-1}(p)$. In other words, with $u = (1, 0, \cdots, 0)$, we have

$$Y(p) = (u, M^{-1}(p)u).$$

Introducing $p = i\lambda$* and the diagonal or symmetric matrix

$$D = \begin{bmatrix} L^{1/2} & 0 \\ 0 & C^{1/2} \end{bmatrix},$$

one can write $M(p)$ in the form

$$M(p) = M(i\lambda) = i \begin{bmatrix} L\lambda & i\gamma \\ -i\gamma^T & C\lambda \end{bmatrix}$$
$$= iD(\lambda I - S)D,$$

where

$$S = \begin{bmatrix} 0 & -i\beta \\ i\beta^T & 0 \end{bmatrix},$$

*Here i denotes $(-1)^{1/2}$.

and $\beta = L^{-1/2}\gamma C^{-1/2}$. It is clear that with $v = D^{-1}u$,

$$Y(p) = (u, M^{-1}(p)u) = \frac{1}{i}(v, (\lambda I - S)^{-1}v).$$

The matrix S is Hermitian, i.e., $S = \bar{S}^T$. This implies that its eigenvalues are real. Choosing λ_k as the distinct values of the eigenvalues, there exist projection matrices P_k such that $P_k = \bar{P}_k^T = P_k^2$ and with which the resolvant $(\lambda I - S)^{-1}$ can be written in the form

$$(\lambda I - S)^{-1} = \sum_k \frac{1}{\lambda - \lambda_k} P_k ,$$

where the sum is taken over the distinct eigenvalues of S. This is an immediate consequence of the spectral theorem for Hermitian matrices (see Halmos [12]). Thus

$$iY(p) = \sum_k \frac{r_k}{\lambda - \lambda_k},$$

where $r_k = (v, P_k v) = | P_k v |^2 \geq 0$ and with $p_k = i\lambda_k$, we have

$$Y(p) = \sum_k \frac{r_k}{p - p_k},$$

which proves that the poles are purely imaginary and simple and the residues r_k are nonnegative.

Since, by definition, $Y(p)$ is real for real p, then if p_k is a pole, $\bar{p}_k$ is a pole also. We can then regroup the terms to get another representation

$$Y(p) = \frac{r_0}{p} + \sum_{\mathrm{Im}\, p_k > 0} \frac{2r_k p}{(p - p_k)(p - \bar{p}_k)} = \frac{r_0}{p} + \sum_{\mathrm{Im}\, p_k > 0} \frac{2r_k p}{p^2 - p_k^2}.$$

Forming the common denominator and writing this expression as a fraction, one recognizes it as identical with that given by Foster.

19. Behavior of $\mathbf{P(i, v)}$ as $| \mathbf{i} | + | \mathbf{v} | \to \infty$. In theorem 3, section 8, it was assumed that the matrix A was positive definite and $B(v) + | \gamma v | \to \infty$ as $| v | \to \infty$. Similar conditions are assumed for theorem 4. If the circuit is complete, these conditions are easily checked through the submatrices α, β, and γ given by (13.2).

From (13.10) and the fact that the resistors in $\tau - \tau'$ are linear, it follows that the matrix A has the form

$$A = \alpha R \alpha^T,$$

where R is a diagonal matrix and the diagonal elements are the resistances of the branches in $\tau - \tau'$. Since α is a $l - l' \times t - t'$ matrix and R a $t - t' \times t - t'$ matrix, then clearly A can be positive definite if and only if the diagonal elements of R are positive and α has rank $l - l'$. This implies, in particular, that $t - t' \geq l - l'$, i.e., the number of resistors in $\tau - \tau'$ must be at least equal to the number of inductors.

In order to evaluate the condition $B(v) + | \gamma v | \to \infty$ as $| v | \to \infty$, we refer to (13.11):

$$B(v) = \int_0^v (\beta^T g(\beta v), dv), \tag{19.1}$$

where $g(\beta v)$ is a column vector with the μth element $g_\mu(w_\mu)$ which is the voltage-current characteristic of the resistor in the μth branch, $\mu \in \mathfrak{L}'$. In section 3 we assumed that a resistor characteristic must be in the first or third quadrant and monotone increasing there if $|v|$ is large enough. A less stringent condition is

$$\frac{g_\mu(w)}{w} \geq \theta > 0 \quad \text{for all} \quad |w| > K, \tag{19.2}$$

where θ and K are positive constants. We shall assume this and show that $B(v) + |\gamma v| \to \infty$ as $|v| \to \infty$ if and only if the combined $l \times t'$ matrix $\begin{pmatrix}\gamma\\ \beta\end{pmatrix}$ has rank t'.

Using (19.1) and (19.2), $B(v)$ can be estimated from below by $\frac{1}{4}\theta\,|\beta v|^2$ if $|v|$ is large enough. Thus

$$\begin{aligned} B(v) + |\gamma v|^2 &\geq \delta(|\beta v|^2 + |\gamma v|^2) \\ &\geq \delta\left(\begin{pmatrix}\gamma\\ \beta\end{pmatrix}v, \begin{pmatrix}\gamma\\ \beta\end{pmatrix}v\right), \end{aligned}$$

where δ is some positive constant. It is clear from the right-hand side that if the matrix $\begin{pmatrix}\gamma\\ \beta\end{pmatrix}$ has rank t', then

$$B(v) + |\gamma v|^2 \geq \delta_1 |v|^2,$$

which proves the "if" part of the statement. On the other hand, if $\begin{pmatrix}\gamma\\ \beta\end{pmatrix}$ has rank less than t', then there exists a vector $v = v_0$, which is a nontrivial solution of the equations

$$\begin{pmatrix}\gamma\\ \beta\end{pmatrix}v = 0.$$

Using the mean value theorem,

$$B(v) + |\gamma v| = \left\{\begin{bmatrix} g(\beta v^*) \\ \dfrac{\gamma v}{|\gamma v|}\end{bmatrix}, \begin{bmatrix}\beta\\ \gamma\end{bmatrix}v\right\},$$

and with $v = \lambda v_0$, we have

$$B(\lambda v_0) + |\lambda\gamma v_0| = 0.$$

Hence, $B(v) + |\gamma v|$ does not approach ∞ for all $|v| \to \infty$, which proves the "only if" part of the statement.

In summary, we have proved that *A is positive definite if and only if α has rank $l - l'$ and R is positive definite, and $B(v) + |\gamma v| \to \infty$ as $|v| \to \infty$ if and only if the matrix $\begin{pmatrix}\gamma\\ \beta\end{pmatrix}$ has rank t'.*

These conditions can be interpreted as follows. Using the notation of section 13, if $v^{(1)}, v^{(3)}, v^{(4)}$ are all zero, i.e., all the voltages outside of $\mathfrak{L} - \mathfrak{L}'$ are zero, then by equations (13.3) we have

$$\alpha v^{(2)} = 0.$$

Thus, if α has rank $l - l'$, then $v^{(2)} = 0$, i.e., all the voltages of $\mathfrak{L} - \mathfrak{L}'$ are also zero.

Similarly, if $v^{(2)}$, $v^{(3)}$, $v^{(4)}$ are all zero, i.e., all the voltages outside of τ' are zero, then equations (13.3) yield

$$-\gamma v^{(1)} = 0,$$
$$-\beta v^{(1)} = 0.$$

Therefore, if the matrix $\begin{pmatrix}\gamma\\ \beta\end{pmatrix}$ has rank t', then $v^{(1)}$ must also vanish, i.e., all the voltages of τ' are zero.

For the conditions of theorem 4 one is led to the requirements that β must have rank t' and $\begin{pmatrix}\gamma^T\\ \alpha^T\end{pmatrix}$ must have rank $l - l'$.

20. Periodic solutions for periodically forced networks. We consider a nonlinear electrical network which contains a time-varying periodic voltage source and, given certain conditions, we shall prove the existence of a periodic solution of the same period. A theorem of this type has been proved by R. Duffin [2] for electrical networks with n degrees of freedom in which he assumed that only the resistors could be nonlinear and that they must be quasi-linear, i.e., the slope of the voltage-current characteristic must be positive everywhere. Duffin also proves the uniqueness of the periodic solution, but this does not hold, in general, for other types of nonlinearities. Levinson [13] proved the existence of a periodic solution for a nonlinear second-order differential equation with a periodic forcing term. The nonlinearities considered in Levinson's paper are more general than those considered here*, but our results apply to systems with n degrees of freedom.

We first consider a network $\mathfrak{N}_0$ which contains no time-varying elements, which is complete, and its mixed potential P_0 is semi-linear of the form

$$P_0(i, v) = -(i, \tfrac{1}{2}Ai + a - \gamma v) + B(v). \tag{20.1}$$

In the theorem to be proved we make assumptions which imply that the network $\mathfrak{N}_0$ is asymptotically stable; essentially, we make the assumptions of theorem 3, section 8. The time-varying network $\mathfrak{N}$ is composed of $\mathfrak{N}_0$ and a periodic voltage source $E(t)$ attached as shown in figure 20, and we want to prove that this circuit has a periodic

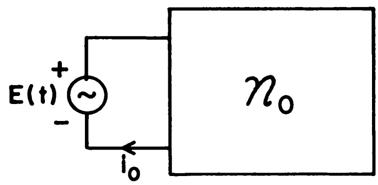

FIG. 20. Periodically excited network.

solution. We also assume that the voltage source is attached so that the current i_0 through it is determined by the set i of the currents through the inductors. Thus, $\mathfrak{N}$ is also complete and semilinear and its mixed potential is

$$\begin{aligned} P(i, v, t) &= P_0(i, v) + E(t)i_0 \\ &= -\tfrac{1}{2}(i, Ai) + (i, \gamma v) - (i, a - e(t)) + B(v), \end{aligned} \tag{20.2}$$

*The nonlinearities considered in Levinson's paper are allowed to be of the form $f(v, dv/dt)$.

where $e(t)$ is a vector with elements $E(t)$ or 0. The system of differential equations to be solved is

$$L(i)\,\frac{di}{dt} = -Ai + \gamma v - a + e(t),$$
$$C(v)\,\frac{dv}{dt} = -\gamma^T i - B_v(v). \tag{20.3}$$

Since $\mathfrak{N}$ is complete, according to the discussion of section 13, we know there exist matrices α and β such that

$$A = \alpha R \alpha^T, \tag{20.4}$$

and

$$B(v) = \int_0^v (\beta^T g(\beta v), dv). \tag{20.5}$$

Here, R is a diagonal matrix with elements R_μ, the resistances of $\mathfrak{N}_i$, and $g(\lambda)$ is a column vector with elements $g_\mu(\lambda_\mu)$, the voltage-current characteristics of the nonlinear resistors in $\mathfrak{N}_v$.

Theorem 13: We assume that

(1) $L(i)$, $C(v)$ are positive definite and their eigenvalues bounded away from zero for all i, v.

(2) The constant matrix R is positive definite.

(3) α has rank $l - l'$ (the number of inductors) and the combined matrix of β and γ has rank t' (the number of capacitors).*

(4) $g_\mu(w)/w \geq \theta > 0$ for all $|w| > K$ with some positive constants θ, K.

(5) $\| L^{1/2}(i) A^{-1} \gamma C^{-1/2}(v) \| \leq 1 - \delta$ with some $\delta > 0$.

If $e(t)$ has period $T > 0$ and is continuously differentiable, then the system (20.3) has at least one periodic solution of period T.

Remark: Condition (3) expresses that sufficiently many resistors are present while (5) imposes a condition on the size of L.

Proof: For the proof we shall construct a closed set D in the phase space with the coordinates $x = (i_1, \cdots, i_r, v_1, \cdots, v_s)$ and $t = 0$ with the following properties:

(a) All solutions with initial values in D will for $t = T$ lie in D again so that the mapping M defined by following the solutions from $t = 0$ (in D) to $t = T$ maps D into itself.

(b) Topologically, D is equivalent to the sphere $|x| \leq 1$. Then it follows from Brouwer's fixed point theorem (see S. Lefshetz [14]) that the mapping M possesses a fixed point in D. The solution initially at this fixed point returns to the fixed point for $t = T$ and is therefore the desired periodic solution.

For the construction of the domain D we use the pair J^*, P^* which was introduced in section 8 (for theorem 3):

$$P^*(x, t) = P(x, t) + (P_i, A^{-1}P_i),$$
$$J^* = \begin{bmatrix} L(i) & 0 \\ -2\gamma^T A^{-1} L(i) & C(v) \end{bmatrix}. \tag{20.6}$$

*With the notation of (13.1), the $l \times t'$ matrix (c_{13}, c_{14}) should have rank t'.

As in section 8, one finds

$$\frac{d}{dt} P^*(x, t) = -\left(\frac{dx}{dt}, J^* \frac{dx}{dt}\right) + \frac{\partial P^*}{\partial t}. \tag{20.7}$$

In case e is time independent, the last term vanishes while the first term is always negative. Our aim now will be to dominate $\partial P^*/\partial t$ by $(dx/dt, J^* dx/dt)$ for sufficiently large $|x|$. The domain D will be defined by

$$P^*(x, 0) \leq p \tag{20.8}$$

for large positive p. We will derive now that the assumptions of theorem 13 imply the properties (a) and (b) of the domain D.

To estimate $\partial P^*/\partial t$ we use (20.6)

$$\frac{\partial P^*}{\partial t} = \frac{\partial P}{\partial t} + 2\left(\frac{\partial^2 P}{\partial t\, \partial i}, A^{-1} \frac{\partial P}{\partial i}\right),$$

and (20.2)

$$\frac{\partial P^*}{\partial t} = \left(\frac{de}{dt}, i + 2A^{-1}[-Ai + \gamma v - a + e(t)]\right).$$

Since de/dt is bounded for all t, we have

$$\left| \frac{\partial P^*}{\partial t} \right| \leq c(| i | + | v | + 1), \tag{20.9}$$

with some positive c. Hence, $\partial P^*/\partial t$ grows at most linearly with $| x |$.

Next it is our aim to show that $(dx/dt, J^* dx/dt)$ grows at least quadratically with $| x |$, thus dominating $\partial P^*/\partial t$. For this purpose it is convenient to introduce new variables y by the linear transformation

$$y = Sx - b, \tag{20.10}$$

where

$$S = \begin{bmatrix} -A & \gamma \\ 0 & I \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a - e(t) \\ 0 \end{bmatrix},$$

which was introduced in section 8. Then $P^*(x, t)$ transforms into

$$Q(y, t) = P^*(x, t) = \tfrac{1}{2}(y_1, A^{-1}y_1) + U(y_2), \tag{20.11}$$

(see 8.10) where

$$y_1 = -Ai + \gamma v - b_1, \qquad y_2 = v,$$

and

$$U(v) = \tfrac{1}{2}((b_1 - \gamma v), A^{-1}(b_1 - \gamma v)) + B(v). \tag{20.12}$$

To prove our statement we will use now the variables y which are related to the x by a nonsingular linear transformation. We shall derive the remaining estimates from the following lemma.

Lemma. With some positive constants $c > 0$, $c_1 > 0$, we have

$$\left(y, \frac{\partial Q}{\partial y}\right) > c | y |^2 \quad \text{for} \quad | y | > c_1 . \tag{20.13}$$

We postpone the proof to the end and first show how theorem 13 follows from this lemma. The left-hand side of (20.13) can be considered as a radial derivative of Q, namely with $|y| = r$, we have

$$r \frac{\partial Q}{\partial r} > cr^2 \quad \text{for} \quad r > c_1 ,$$

or

$$\left| \frac{\partial Q}{\partial y} \right| \geq \frac{\partial Q}{\partial r} > c\,|y| \quad \text{for} \quad |y| > c_1 , \tag{20.14}$$

and integrating,

$$|Q| \geq \frac{c}{2} |y|^2 - c_2 \quad \text{for} \quad |y| > c_1 . \tag{20.15}$$

To estimate the right-hand side of (20.7), we use that $P_x^* = S^T Q_y$ and the consequence of assumptions (1) and (5):

$$\left(\frac{dx}{dt} , J^* \frac{dx}{dt}\right) = (J^{*-1} P_x^* , P_x^*) \geq \delta_1 \, |P_x^*|^2 \geq \delta_2 \, |Q_y|^2,$$

and because of (20.14)

$$\left(\frac{dx}{dt} , J^* \frac{dx}{dt}\right) \geq \delta_3 \, |y|^2 \quad \text{for} \quad |y| \geq c_3 .$$

From (20.7) and (20.9) it follows

$$\frac{d}{dt} Q(y, t) = -\left(\frac{dx}{dt} , J^* \frac{dx}{dt}\right) + \frac{\partial Q}{\partial t} \leq -\frac{\delta_3}{2} |y|^2 \quad \text{for} \quad |y| \geq c_3 . \tag{20.16}$$

For the construction of the domain D', defined by

$$Q(y, 0) \leq p,$$

we choose p so large that the sphere $|y| \leq c_3$ is contained in D'. This is certainly possible since Q is continuous for all y and tends to ∞ only as $|y| \to \infty$. To show that D' is topologically equivalent to the sphere $|x| \leq 1$, we observe that D' is starlike: for every ray $y = r\eta$ (with a fixed η, $|\eta| = 1$), the function $Q(y, 0)$ is by the lemma a monotone increasing function of r for sufficiently large r. Since also $Q \to \infty$ as $r \to \infty$ (by (20.15)) it follows that the boundary $Q = p$ of D' intersects every ray at exactly one point $y = R(\eta)\eta$ (for sufficiently large p). Therefore, D' can be represented in the form

$$|y| \leq R(\eta),$$

where $R(\eta)$ is a continuous function of η and, say, $R(\eta) > 1$. This set is topologically equivalent to a sphere since one can just map D' into $|x| \leq 1$ by the continuous mapping which takes $y = r\eta$ into $x = (r/R(\eta))\eta$.

Let $y(t)$ be any solution for which the initial values belong to the boundary of D'. Then $Q(y(t), t)$ decreases (by (20.16)) as long as $|y(t)| > c_3$ and, since $Q(y, T) = Q(y, 0)$, then $y(T)$ belongs to D'. Therefore, the mapping $y(0) \to y(T)$ takes the boundary of D' into D' and, since D' is topologically equivalent to a sphere and the mapping continuous, one-one, D' is mapped into D'.

Applying Brouwer's fixed point theorem to the mapping $y(0) \to y(T)$ defined in D', we find a fixed point $y^*(0)$ in D' and thus a periodic solution $y^*(t)$ with period T.

It remains to prove the lemma which, by (20.11), takes the form

$$(y_1, A^{-1}y_1) + (y_2, U_{v_2}) > c \mid y \mid^2 \quad \text{for} \quad \mid y \mid > c_1 .$$

Since A^{-1} is positive definite (see (20.4) and assumption (3)), it suffices to prove

$$(v, U_v) > c \mid v \mid^2 \quad \text{for large} \quad \mid v \mid. \tag{20.17}$$

From (20.12) we find

$$(v, U_v) \geq (\gamma v, A^{-1}\gamma v) - \delta_1 \mid v \mid + (v, B_v)$$
$$\geq \delta \mid \gamma v \mid^2 + (v, B_v) - \delta_1 \mid v \mid \quad \text{for large} \quad \mid v \mid,$$

where $\delta > 0$. Using (20.5), we have

$$(v, B_v) = (\beta v, g(\beta v)) = \sum_\mu w_\mu g_\mu(w_\mu),$$

where $w = \beta v$. By assumption (4) this can be estimated by $\frac{1}{2}\theta \sum_\mu w_\mu^2$ from below, and we find

$$(v, U_v) \geq \delta_2(\mid \gamma v \mid^2 + \mid \beta v \mid^2) - \delta_1 \mid v \mid \quad \text{for} \quad \mid v \mid > c_4 .$$

Since by assumption (3) $\gamma v = \beta v = 0$ implies $v = 0$, the estimate (20.14) follows. This completes the proof of the lemma and, hence, the proof of theorem 13.

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