UNIFIED FORMALISM OF THE LINEARIZED COMPRESSIBLE FLOW FIELDS*

BY
IWAO HOSOKAWA**
Brown University

1. Introduction. The steady motion of an inviscid, non-conducting, compressible fluid is governed by a complicated nonlinear equation. The difficulty of solving this equation has led many authors to the extensive use of a linear approximation to the original equation based on the assumption of a small disturbance in the flow field, in particular in application to aeronautics (see Ward [1]). The flow field given by such a conventional linearization, however, has a divergent discontinuity at a free stream Mach number of $M_\infty = 1$. Therefore, any successive approximation method starting from the linear equation is not uniformly convergent at the point $M_\infty = 1$. In order to avoid this difficulty, the so-called transonic approximation to the flow equation was proposed by Oswatitsch, von Kármán [2] and others. This approximation retains one of the nonlinear terms of the original equation in such a way that the solution is the first approximation to the compressible flow field and is no longer divergent for $M_\infty = 1$, but the nonlinearity so retained is obviously very inconvenient for general analysis. For this reason, the linearized transonic flow theory has been developed by Oswatitsch and Keune [3] and Maeder and Thommen [4]. This linearized theory simplifies the transonic approximation by linearizing the nonlinear term in an appropriate way (see (2.1)). It may be said that this theory succeeds at least in achieving the continuity of the flow field transition through $M_\infty = 1$, though it is questionable whether it is really the first approximation to the original flow field. The author's recent approach [5], however, shows that based on this linearized transonic flow field, the equation of the transonic approximation can be solved at least in the neighborhood of a suitable class of thin bodies. This solution is in the form of the linearized transonic flow field plus a compensation term which embodies the influence of the nonlinearity.†

It seems worthwhile to prepare the solution for the linearized transonic flow field in a general form as was done by Ward [1] and others for the linear sub- and supersonic theories. Such a general solution, on the other hand, holds a new mathematical interest; it is obviously desirable to treat the sub- (elliptic) and supersonic (hyperbolic) flow fields simultaneously in a unified manner including the case of $M_\infty = 1$. There has been some trouble in completing the parallelism of the two fields, or unifying them, because special accounts about Hadamard's finite integral, the Mach cone, etc. are needed in

*Received February 25, 1963; revised manuscript received September 30, 1963. The results in this paper were obtained in the course of research sponsored by the Air Force Office of Scientific Research under Grant AF-169-63.

**On leave from National Aeronautical Laboratory, Tokyo, Japan.

†This theory of refinement of transonic linearization was independently derived by Maeder and Thommen [6] in a different way. On the other hand, Spreiter [7] developed the local linearization method which may be considered as a modification of the linearized transonic flow theory and can give good approximate solutions for shock-less transonic flow over a body.

the supersonic case. Nevertheless, the above desire is realized if the fields are described in terms of *distributions* which are developed here. Such a unified formalism is useful for understanding the separate knowledges of sub- and supersonic theories of wings and bodies. The method of distributions was first applied by Dorfner [8] to the three-dimensional supersonic problem of gas dynamics. In this paper, the original representation of distributions by Schwartz [9] is used for a clear understanding and to avoid any confusion of distributions with functions. As an example, the flow around a thin wing is dealt with in some detail, and some new results on the lifting problem are added.

2. Basic equation. The basic equation of the linearized transonic flow theory is written as follows:

$$(1 - M_\infty^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = K\varphi_x , \qquad K > 0, \tag{2.1}$$

where φ is the disturbance velocity potential normalized by the free stream velocity U_∞ , K is a certain constant for the transonic linearization and x is taken in the direction of the free stream, the subscripts x, y, z indicating differentiation with respect to the coordinates. This equation is well known as a convenient simplification of the so-called transonic approximation to the flow equation

$$(1 - M_\infty^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = (\gamma + 1)M_\infty^2\varphi_x\varphi_{xx} \tag{2.2}$$

(see [10]), where γ is the ratio of specific heats. The terms on the right-hand side in the above equations will tend to vanish in comparison with the other terms as M_∞ increases or decreases from unity, so that we have the conventional linear theory in the limits. Equation (2.1) may therefore be called the extensively linearized compressible flow equation. As regards discussions of possible refinement of the approximation and how to deal with the constant K, the reader is referred to [5], [6] and [7].

For convenience of notation, the following new quantities are introduced:

$$\beta^2 = 1 - M_\infty^2 = -m^2, \qquad \alpha = K/2\beta^2, \tag{2.3}$$

$$\xi = x, \qquad \eta = \beta y, \qquad \zeta = \beta z, \tag{2.4}$$

$$\varphi_0(\xi, \eta, \zeta) = e^{-\alpha\xi}\varphi(\xi, \eta, \zeta). \tag{2.5}$$

Equation (2.1) thus reduces to

$$(\Delta_L - \alpha^2)\varphi_0 = 0, \tag{2.6}$$

where Δ_L is the Laplacian in the (ξ, η, ζ)-space.

Since η and ζ are imaginary for $M_\infty > 1$, the quantities ξ, η, ζ constitute a pseudo-Euclidean space. The effect of the imaginary quantities reveals itself in a scalar invariant, as is seen for

$$(\xi^2 + \eta^2 + \zeta^2)^{1/2} = (x^2 - (M_\infty^2 - 1)y^2 - (M_\infty^2 - 1)z^2)^{1/2} \tag{2.7}$$

which is, in fact, a Minkowski-type distance. The case $M_\infty = 1$ gives rise to $\alpha = \pm\infty$, and should therefore be excluded, but the solution for φ at $M_\infty = 1$ can be obtained successfully by a limiting process afterwards. In case $K = 0$, the equation is reduced to the conventional linear theory.

In case $K \neq 0$, however, a new feature different from the former appears. The velocity potential caused by a vortex distribution cannot be obtained explicitly as easily as in the former case. This fact makes it impossible to treat the problem of lift conveniently by the superposition of vortex-lines or -sheets, as in the lifting line theory, and the flow reversal theorem cannot be derived. As a result, the most successful way to complete the solution subject to each boundary condition for all M_∞ seems to consist in the application of the extended Green's theorem, which may be understood as the method of sources and doublets and is most naturally and inclusively formulated by the use of distributions.

3. Unified general solution. First, the extension of the field to a distribution (see Appendix) is made as follows:

$$\boldsymbol{\varphi} = \varphi \mathbf{H}, \tag{3.1}$$

where the function $H(\xi, \eta, \zeta)$ corresponding to $\mathbf{H}$ is defined as unity inside the boundary (within the flow field) and zero elsewhere, and φ is redefined here to be a regular function in the whole space. Multiplication by $e^{-\alpha\xi}$ yields

$$\boldsymbol{\varphi}_0 = \varphi_0 \mathbf{H}. \tag{3.2}$$

Next, application of the operator $(\Delta_L - \alpha^2)$ to (3.2) furnishes

$$(\Delta_L - \alpha^2)\boldsymbol{\varphi}_0 = [(\Delta_L - \alpha^2)\varphi_0]\mathbf{H} + \sum_{\xi,\eta,\zeta} \frac{\partial \varphi_0}{\partial \xi} \frac{\partial}{\partial \xi} \mathbf{H} + \sum_{\xi,\eta,\zeta} \frac{\partial}{\partial \xi}\left[\varphi_0 \frac{\partial}{\partial \xi} \mathbf{H}\right] \tag{3.3}$$

(see (A14)). Since the first term of the right-hand side vanishes, (3.3) reduces to

$$(\Delta_L - \alpha^2)\boldsymbol{\varphi}_0 = \mathbf{G} \tag{3.4}$$

$$\mathbf{G} = \sum_{\xi,\eta,\zeta,i} (\pm)_i \left[\frac{\partial \varphi_0}{\partial \xi_{(\xi=\xi_{0i})}} \boldsymbol{\delta}_{\xi,\xi_{0i}} + \varphi_{0(\xi=\xi_{0i})} \frac{\partial}{\partial \xi} \boldsymbol{\delta}_{\xi,\xi_{0i}}\right]. \tag{3.5}$$

Here $\boldsymbol{\delta}_{\xi,\xi_{0i}}$ corresponds to the so-called delta function $\delta(\xi - \xi_0)$. Similarly $\boldsymbol{\delta}_{\xi\eta\zeta}$ corresponds to $\delta(\xi)\delta(\eta)\delta(\zeta)$ (see (A3)–(A6), (A8), (A11)–(A13)).

In order to obtain the explicit expression of $\boldsymbol{\varphi}_0$, it is useful to know the fundamental distribution $\mathbf{f}$ which satisfies

$$(\Delta_L - \alpha^2)\mathbf{f} = \boldsymbol{\delta}_{\xi\eta\zeta}\ . \tag{3.6}$$

The function corresponding to $\mathbf{f}$ is given as

$$f(\xi, \eta, \zeta) = -e^{-\alpha\rho}/4\pi\rho \qquad \text{for} \quad M_\infty < 1 \tag{3.7}$$

$$= \begin{cases} -\cosh(\alpha\rho)/2\pi\rho, & \xi > [-(\eta^2 + \zeta^2)]^{1/2} \\ 0, & \text{elsewhere,} \end{cases} \qquad \text{for} \quad M_\infty > 1 \tag{3.8}$$

where $\rho = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$. The proof of these consists in showing that

$$\iiint_{-\infty}^{\infty} f(\xi, \eta, \zeta)(\Delta_L - \alpha^2)\phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta = \phi(0, 0, 0),$$

because (3.6) leads to $\mathbf{f}[(\Delta_L - \alpha^2)\phi] = \boldsymbol{\delta}_{\xi\eta\zeta}$ (see (A2), (A9)).

Now, making a convolution between (3.6) and $\mathbf{G}$, we have

$$(\Delta_L - \alpha^2)\mathbf{f} * \mathbf{G} = \boldsymbol{\delta}_{\xi\eta\zeta} * \mathbf{G} = \mathbf{G} \tag{3.9}$$

(see A16)). Comparison between (3.4) and (3.9) thus results in

$$\boldsymbol{\varphi}_0 = \mathbf{f} * \mathbf{G}. \tag{3.10}$$

(No non-vanishing solution exists for a homogeneous linear equation of a distribution.) This corresponds to what is called the extended Green's theorem. If we take account of (A18) and (A19) in the Appendix, the function corresponding to (3.10) inside the boundary becomes

$$\varphi_0(\xi', \eta', \zeta') = \int_B \frac{\partial \varphi_0}{\partial \nu}(\xi, \eta, \zeta) f(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS - \int_B \varphi_0(\xi, \eta, \zeta) \frac{\partial f}{\partial \nu}(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS, \tag{3.11}$$

where $\int_B dS$ indicates the surface integral taken over all the boundaries and $\partial/\partial\nu$ is the normal derivative at the surface element of the boundaries in the (ξ, η, ζ)-space directed towards the interior of the flow field. Outside the boundary, we have

$$0 = \int_B \frac{\partial \varphi_0}{\partial \nu}(\xi, \eta, \zeta) f(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS - \int_B \varphi_0(\xi, \eta, \zeta) \frac{\partial f}{\partial \nu}(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS, \tag{3.12}$$

which shows that there must in general be some relation between the source $(\partial\varphi_0/\partial\nu)$ and the doublet (φ_0).

Returning to the field φ, we obtain the relation

$$\varphi = e^{\alpha\xi}(\mathbf{f} * \mathbf{G}) \tag{3.13}$$

$$= (e^{\alpha\xi}\mathbf{f}) * (e^{\alpha\xi}\mathbf{G}). \tag{3.14}$$

Its formal interpretation of the same kind as (3.11) is

$$\varphi(\xi', \eta', \zeta') = \int_B \frac{\partial \varphi}{\partial \nu}(\xi, \eta, \zeta) e^{\alpha(\xi' - \xi)} f(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS - \int_B \varphi(\xi, \eta, \zeta) e^{\alpha(\xi' - \xi)} \frac{\partial f}{\partial \nu}(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\, dS. \tag{3.15}$$

The scalar product $(\partial/\partial\nu) \cdot dS$ is always a real value. For the case of $M_\infty > 1$, it is nothing more than the product of the surface element and the conormal derivative there in the real space, and the domain of integration is contained within the upstream-Mach cone as a result of (3.8).

In the limit, when M_∞ is equal to unity, we have

$$e^{\alpha\xi} f(\xi, \eta, \zeta) = \begin{cases} -\dfrac{1}{4\pi} \dfrac{\exp\left[-\alpha(\eta^2 + \zeta^2)/2\xi\right]}{\xi}, & \xi > 0, \\ 0, & \xi < 0, \end{cases} \tag{3.16}$$

and

$$\alpha(\eta^2 + \zeta^2) = K(y^2 + z^2)/2 \tag{3.17}$$

from each expression of (3.7), (3.8). This insures continuity of the solution with respect to M_∞ at $M_\infty = 1$. Thus, (3.13) through (3.15) give a unified solution of (2.2).

The source and doublet distributions are decided subject to the boundary condition physically set in each case. An example of the application to a special boundary condition is given in the next section.

4. Solution for a thin wing. Since the location of the boundary can be approximated by a plan-form included in the plane $z = 0$ in the small perturbation treatment, (3.5) is reduced to

$$\mathbf{G} = \left(\frac{\partial\varphi_0}{\partial\zeta_{(\zeta=+0)}} - \frac{\partial\varphi_0}{\partial\zeta_{(\zeta=-0)}}\right)\boldsymbol{\delta}_\zeta + (\varphi_{0(\zeta=+0)} - \varphi_{0(\zeta=-0)})\frac{\partial}{\partial\zeta}\boldsymbol{\delta}_\zeta . \tag{4.1}$$

Hence (3.13) and (3.14) become

$$\varphi = e^{\alpha\xi}\left(\Delta\frac{\partial\varphi_0}{\partial\zeta}\boldsymbol{\delta}_\zeta*\mathbf{f} + \Delta\varphi_0\boldsymbol{\delta}_\zeta*\frac{\partial}{\partial\zeta}\mathbf{f}\right) \tag{4.2}$$

$$= \Delta\frac{\partial\varphi}{\partial\zeta}\boldsymbol{\delta}_\zeta*e^{\alpha\xi}\mathbf{f} + \Delta\varphi\boldsymbol{\delta}_\zeta*e^{\alpha\xi}\frac{\partial}{\partial\zeta}\mathbf{f}, \tag{4.3}$$

where Δ denotes the difference of the subsequent quantity between the two surfaces of the wing plane at $\zeta = 0$. Since f is symmetric with respect to this plane, the first and second terms in (4.3) are the symmetric and antisymmetric parts of φ, respectively.

Symmetric Problem. If we denote the boundary of the wing as

$$Z = (\pm 1/2)Z_s(x, y) + Z_a(x, y), \tag{4.4}$$

where the double sign corresponds to the upper and lower surfaces, and apply the linearized boundary condition

$$\frac{\partial Z}{\partial x} = \lim_{z\to 0}\frac{\partial\varphi}{\partial z} \tag{4.5}$$

on the respective surfaces, the symmetric term of φ is found as

$$\varphi_{(s)} = \frac{1}{\beta}\frac{\partial Z_s}{\partial x}\boldsymbol{\delta}_\zeta*e^{\alpha\xi}\mathbf{f}. \tag{4.6}$$

Indeed, with the aid of (3.7), (3.8) and (3.16),

$$\varphi_{(s)} = -\frac{1}{4\pi}\iint_{\text{Wing}}\frac{\partial Z_s}{\partial x}\frac{e^{\alpha(x'-x)}\exp\{-\alpha[(x'-x)^2+\beta^2(y'-y)+\beta^2z'^2]^{1/2}\}}{[(x'-x)^2+\beta^2(y'-y)^2+\beta^2z'^2]^{1/2}}\,dx\,dy \quad \text{for } M_\infty \leqq 1, \tag{4.7}$$

$$= -\frac{1}{2\pi}\iint_{\text{Wing},\, x<x'-m[(y'-y)^2+z'^2]^{1/2}}\frac{\partial Z_s}{\partial x}\cdot\frac{e^{\alpha(x'-x)}\cosh\{\alpha[(x'-x)^2-m^2(y'-y)^2-m^2z'^2]^{1/2}\}}{[(x'-x)^2-m^2(y'-y)^2-m^2z'^2]^{1/2}}\,dx\,dy \quad \text{for } M_\infty \geqq 1, \tag{4.8}$$

$$= -\frac{1}{4\pi}\iint_{\text{Wing},\, x<x'}\frac{\partial Z_s}{\partial x}\frac{\exp\{-K[(y'-y)^2+z'^2]/4(x'-x)\}}{x'-x}\,dx\,dy \quad \text{for } M_\infty = 1. \tag{4.9}$$

The corresponding solution for a two-dimensional airfoil is readily derived by integration of (4.7) through (4.9) with respect to y. Thus,

$$\varphi_{(s)} = -\frac{1}{2\pi\beta}\int_0^1\frac{dZ_s}{dx}e^{\alpha(x'-x)}K_0\{\alpha[(x'-x)^2+\beta^2z'^2]^{1/2}\}\,dx, \quad M_\infty \leqq 1, \tag{4.10}$$

$$= -\frac{1}{2m}\int_0^{x'-m|z'|}\frac{dZ_s}{dx}e^{\alpha(x'-x)}I_0\{|\alpha|\,[(x'-x)^2-m^2z'^2]^{1/2}\}\,dx, \quad M_\infty \geqq 1, \tag{4.11}$$

$$= -\frac{1}{2(\pi K)^{1/2}}\int_0^{x'}\frac{dZ_s}{dx}\frac{\exp[-Kz'^2/4(x'-x)]}{(x'-x)^{1/2}}\,dx, \qquad M_\infty = 1, \tag{4.12}$$

where K_0 and I_0 are modified Bessel functions of the second and first kinds respectively, the chord-length of the airfoil is taken the unity of length, and the leading edge is located at the origin.

Antisymmetric Problem. The main work in this problem is to determine the unknown $\Delta\varphi$ on the basis of the boundary condition.

First, differentiating the second term in (4.3) with respect to ζ,

$$\frac{\partial}{\partial\zeta}\varphi_{(a)} = \Delta\varphi\boldsymbol{\delta}_\zeta * e^{\alpha\xi}\frac{\partial^2}{\partial\zeta^2}\mathbf{f} \tag{4.13}$$

By use of (3.6) and the formulas of distribution, (A14) and (A17), this is reduced to

$$\begin{aligned}\frac{\partial}{\partial\zeta}\varphi_{(a)} &= \Delta\varphi\boldsymbol{\delta}_\zeta * e^{\alpha\xi}\left[\left(\alpha^2 - \frac{\partial^2}{\partial\xi^2} - \frac{\partial^2}{\partial\eta^2}\right)\mathbf{f} + \boldsymbol{\delta}_{\xi\eta\zeta}\right] \\ &= \frac{\partial\Delta\varphi}{\partial\xi}\boldsymbol{\delta}_\zeta * \left(-\frac{\partial}{\partial\xi} + 2\alpha\right)e^{\alpha\xi}\mathbf{f} - \frac{\partial\Delta\varphi}{\partial\eta}\boldsymbol{\delta}_\zeta * e^{\alpha\xi}\frac{\partial}{\partial\eta}\mathbf{f} + \Delta\varphi\boldsymbol{\delta}_\zeta\,.\end{aligned} \tag{4.14}$$

Hence, on account of the boundary condition (4.5) applied to the antisymmetric part, we have an integral equation for $\Delta\varphi$:

$$\begin{aligned}\frac{\partial Z_a}{\partial x'} = \beta^2 &\iint_{\text{Wing}} \frac{\partial\Delta\varphi}{\partial x}\left(\frac{\partial}{\partial x} + 2\alpha\right)e^{\alpha(x'-x)}f(\xi' - \xi,\, \eta' - \eta,\, 0)\, dx\, dy \\ &+ \iint_{\text{Wing+Wake}} \frac{\partial\Delta\varphi}{\partial y}\, e^{\alpha(x'-x)}\frac{\partial}{\partial y}f(\xi' - \xi,\, \eta' - \eta,\, 0)\, dx\, dy.\end{aligned} \tag{4.15}$$

(Since $z \neq 0$ in the limiting process $z \to 0$, the last term in (4.14) does not contribute to (4.15). No account has been taken of the discontinuity of $\partial\varphi/\partial x$ on the wake since the pressure is continuous there.)

On the other hand, let us notice the interesting fact that we can have another convenient equation by introducing a new distribution $\mathbf{h}$ such that

$$e^{\alpha\xi}\frac{\partial^2}{\partial\zeta^2}\mathbf{f} = \frac{\partial}{\partial\xi}\mathbf{h}, \tag{4.16}$$

and consequently,

$$\frac{\partial}{\partial\zeta}\varphi_{(a)} = \frac{\partial\Delta\varphi}{\partial\xi}\boldsymbol{\delta}_\zeta * \mathbf{h}. \tag{4.17}$$

Hence, another integral equation,

$$\frac{\partial Z_a}{\partial x'} = \beta^2 \iint_{\text{Wing}} \frac{\partial\Delta\varphi}{\partial x}\, h(\xi' - \xi,\, \eta' - \eta,\, 0)\, dx\, dy, \tag{4.18}$$

is established in place of (4.15). Here $h(\xi, \eta, 0)$ is easily given as

$$h(\xi, \eta, 0) = \frac{1}{4\pi}\frac{1}{\beta^2 y^2}\frac{e^{\alpha[x-(x^2+\beta^2y^2)]^{1/2}}}{(x^2 + \beta^2y^2)^{1/2}}[x + (x^2 + \beta^2y^2)]^{1/2}, \quad \text{for} \quad M_\infty \leqq 1 \tag{4.19}$$

$$= \begin{cases} \dfrac{1}{2\pi}\dfrac{e^{\alpha x}}{\beta^2y^2}\dfrac{e^{\alpha(x^2-m^2y^2)^{1/2}}[x-(x^2-m^2y^2)^{1/2}]+e^{-\alpha(x^2-m^2y^2)^{1/2}}[x+(x^2-m^2y^2)^{1/2}]}{2(x^2-m^2y^2)^{1/2}}, & x > m\,|y| \\ 0, & x < m\,|y| \end{cases} \quad \text{for} \quad M_\infty \geqq 1. \tag{4.20}$$

(Start from the identity relation: $-2\sigma(\partial/\partial\sigma)e^{\alpha(\xi\mp\rho)}/\rho = (\partial/\partial\xi)e^{\alpha(\xi\mp\rho)}(\xi \pm \rho)/\rho$ where $\sigma = \eta^2 + \zeta^2$.)

Equation (4.18) is simpler than (4.15) since the former has no surface integral over the wake. However, a further general analysis of these is beyond the scope of the present formalism. In fact, even in the simplest case of incompressible flow it is not easy to solve (4.15) or (4.18) without using Jones' approximate method which assumes a suitable expansion and determines its coefficients. Equation (4.15) is essentially a direct extension of the integral equation given for such a case by Robinson and Laurmann [11] for the general case of compressible flow.

In the two-dimensional case, we can carry out a far simpler analysis in order to get the following integral equations for the lift distribution:

$$\frac{dZ_a}{dx'} = -\frac{\alpha\beta}{2\pi}\int_0^1 \frac{d\Delta\varphi}{dx} e^{\alpha(x'-x)}\{K_0[\alpha\,|x'-x|] + K_1[\alpha(x'-x)]\}\,dx \quad \text{for} \quad M_\infty \leqq 1 \tag{4.21}$$

$$= \alpha m \int_0^{x'} \frac{d\Delta\varphi}{dx} e^{\alpha(x'-x)} I_0[|\alpha|\,(x'-x)]\,dx - \frac{m}{2}\int_0^{x'} \frac{d^2\Delta\varphi}{dx^2} e^{\alpha(x'-x)} I_0[|\alpha|\,(x'-x)]\,dx$$

$$- \frac{m}{2}\frac{d\Delta\varphi}{dx'_{(x'=+0)}} e^{\alpha x'} I_0[|\alpha|\,x'] \quad \text{for} \quad M_\infty \geqq 1. \tag{4.22}$$

These are derived directly from the following modification of (4.14):

$$\frac{\partial}{\partial\zeta}\varphi_{(a)} = \frac{d\Delta\varphi}{dx}\,\boldsymbol{\delta}_\zeta * e^{\alpha\xi}\left(\alpha - \frac{\partial}{\partial\xi}\right)\mathbf{f} + \Delta\varphi\boldsymbol{\delta}_\zeta \tag{4.25}$$

$$= \left[\left(2\alpha\frac{d\Delta\varphi}{dx} - \frac{d^2\Delta\varphi}{dx^2}\right)\boldsymbol{\delta}_\zeta - \sum_i \Delta_{Ei}\frac{d\Delta\varphi}{dx}\,\boldsymbol{\delta}_{\xi\zeta,\xi_{Ei}}\right] * e^{\alpha\xi}\mathbf{f} + \Delta\varphi\boldsymbol{\delta}_\zeta\,, \tag{4.26}$$

where Δ_{Ei} denotes a discontinuous jump, if any, of $d\Delta\varphi/dx$ at the edge of the airfoil $x = x_{Ei}$.

It is reasonable to assume that (4.21) should be associated with the Kutta-Joukowski condition at the trailing edge, $d\Delta\varphi/dx(1) \neq \infty$, as we did in the conventional linear theory ($\alpha = 0$). Thus, this can be uniquely solved by successive approximations according to Muskhelishvili's method [12] for a singular integral equation. On the other hand, (4.22) can be solved by the Laplace transformation with ease. In particular, we have

$$\frac{dZ_a}{dx'} = -\frac{1}{2}\left(\frac{K}{\pi}\right)^{1/2}\int_0^{x'} \frac{d\Delta\varphi}{dx}\frac{dx}{(x'-x)^{1/2}} \quad \text{for} \quad M_\infty = 1. \tag{4.23}$$

which is a well-known Abel integral equation.

5. Conclusion. Within the frame of the linearized transonic flow theory, the velocity potential of compressible flow was formulated in a unified manner. The principal formalism was conveniently given in terms of distributions. As an example, the flow around a thin wing was analyzed, and a general method of solving the lifting problem was considered. The obtained formulas completely cover the conventional sub- and supersonic linear theories in both limits $\alpha = \pm 0$. Similar formulations should be possible for the flow around a slender body. In conclusion the writer would like to express many thanks to Professor P. F. Maeder of Brown University for his encouragement and support of this work and to Professor W. Prager of IBM Research Laboratory Zürich for his advice concerning the presentation of the material.

Appendix

Notations and Formulas Concerning Distributions

Distributions are denoted by bold face characters.

A1. $$\mathbf{F}(\phi) = \iiint_{-\infty}^{\infty} F(\xi, \eta, \zeta)\phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

ϕ is a so-called testing function.

A2. $$\frac{\partial}{\partial \xi} \mathbf{F}(\phi) = -\mathbf{F}\left(\frac{\partial \phi}{\partial \xi}\right)$$

A3. $$\boldsymbol{\delta}_{\xi\eta\zeta}(\phi) = \phi(0, 0, 0) = \iiint_{-\infty}^{\infty} \delta(\xi, \eta, \zeta)\phi(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

A4. $$\boldsymbol{\delta}_{\xi\eta\zeta,\xi_0\eta_0\zeta_0}(\phi) = \phi(\xi_0, \eta_0, \zeta_0)$$

A5. $$\boldsymbol{\delta}_{\xi\eta,\xi_0\eta_0}(\phi) = \int_{-\infty}^{\infty} \phi(\xi_0, \eta_0, \zeta)\, d\zeta$$

A6. $$\boldsymbol{\delta}_{\xi,\xi_0}(\phi) = \iint_{-\infty}^{\infty} \phi(\xi_0, \eta, \zeta)\, d\eta\, d\zeta$$

A7. $$\boldsymbol{\delta}_{\xi,a\eta}(\phi) = \iint_{-\infty}^{\infty} \phi(a\eta, \eta, \zeta)\, d\eta\, d\zeta$$

$$= \frac{1}{a} \iint_{-\infty}^{\infty} \phi(\xi, \xi/a, \zeta)\, d\xi\, d\zeta = \frac{1}{a}\, \boldsymbol{\delta}_{\eta,\xi/a}$$

A8. $$\frac{\partial}{\partial \xi} \mathbf{H} = \sum_i (\pm)_i \boldsymbol{\delta}_{\xi,\xi_{0i}}$$

H is a three-dimensional unit function which vanishes except in a certain volume of space; ξ_{0i} denotes the ξ coordinate of the i-th boundary point for definite values of η and ζ, where there is a unit step in H; $(\pm)_i$ indicates $+$ for a positive step (from zero to unity).

A9. $$\tau(\xi, \eta, \zeta)\mathbf{F}(\phi) = \mathbf{F}(\tau\phi)$$

A10. $$\frac{\partial}{\partial \xi} \mathbf{F} = \Delta F_0 \boldsymbol{\delta}_{\xi,\xi_0} + \frac{\boldsymbol{\partial} \mathbf{F}}{\boldsymbol{\partial \xi}}$$

ΔF_0 is a jump of F at $\xi = \xi_0(\eta, \zeta)$; $\boldsymbol{\partial}\mathbf{F}/\boldsymbol{\partial\xi}$ is a distribution corresponding to $\partial F/\partial \xi$.

A11. $$\tau(\xi, \eta, \zeta)\boldsymbol{\delta}_{\xi\eta\zeta,\xi_0\eta_0\zeta_0} = \tau(\xi_0, \eta_0, \zeta_0)\boldsymbol{\delta}_{\xi\eta\zeta,\xi_0\eta_0\zeta_0}$$

A12. $$\tau(\xi, \eta, \zeta)\boldsymbol{\delta}_{\xi\eta,\xi_0\eta_0} = \tau(\xi_0, \eta_0, \zeta)\boldsymbol{\delta}_{\xi\eta,\xi_0\eta_0}$$

A13. $$\tau(\xi, \eta, \zeta)\boldsymbol{\delta}_{\xi,\xi_0} = \tau(\xi_0, \eta, \zeta)\boldsymbol{\delta}_{\xi,\xi_0}$$

A14. $\frac{\partial}{\partial \xi}(\tau \mathbf{F}) = \frac{\partial \tau}{\partial \xi}\mathbf{F} + \tau \frac{\partial}{\partial \xi}\mathbf{F} + \Delta\tau_0 F(\xi_0, \eta, \zeta)\boldsymbol{\delta}_{\xi,\xi_0}$

$\Delta\tau_0$ is a jump of τ at $\xi = \xi_0$.

A15. $\mathbf{F}*\mathbf{G} = \iiint (F*G)\phi \, d\xi \, d\eta \, d\zeta$

* denotes a convolution product.

A16. $\boldsymbol{\delta}_{\xi\eta\zeta}*\mathbf{F} = \mathbf{F}$

A17. $$\frac{\partial}{\partial \xi}\mathbf{F}*\mathbf{G} = \left(\frac{\partial}{\partial \xi}\mathbf{F}\right)*\mathbf{G} = \mathbf{F}*\left(\frac{\partial}{\partial \xi}\mathbf{G}\right) = \Delta F_0 \boldsymbol{\delta}_{\xi,\xi_0}*\mathbf{G} + \frac{\partial \mathbf{F}}{\partial \xi}*\mathbf{G} = \iiint_{-\infty}^{\infty} \phi \, d\xi \, d\eta \, d\zeta$$
$$\times \iiint G(\xi - \xi', \eta - \eta', \zeta - \zeta') \, dF(\xi'; \eta', \zeta') \, d\eta' \, d\zeta' \text{ (Stieltjes integral)}$$

A18. $$\sum_{\xi,\eta,\zeta} \mathbf{F}*(\pm)_i \frac{\partial \tau}{\partial \xi_{(\xi-\xi_{0i})}} \boldsymbol{\delta}_{\xi,\xi_{0i}} = \iiint_{-\infty}^{\infty} d\xi' \, d\eta' \, d\zeta' \, \phi(\xi', \eta', \zeta') \sum_{\xi,\eta,\zeta,i} (\pm)_i$$
$$\cdot \iint F(\xi' - \xi_{0i}, \eta' - \eta, \zeta' - \zeta) \frac{\partial \tau}{\partial \xi_{(\xi-\xi_{0i})}} \, d\eta \, d\zeta$$
$$= \iiint_{-\infty}^{\infty} d\xi' \, d\eta' \, d\zeta' \, \phi(\xi', \eta', \zeta') \int_B F(\xi' - \xi, \eta' - \eta, \zeta' - \zeta) \frac{\partial \tau(\xi, \eta, \zeta)}{\partial \nu} \, dS$$

A19. $$\sum_{\xi,\eta,\zeta,i} \frac{\partial}{\partial \xi} \mathbf{F}*(\pm)_i \tau_{(\xi-\xi_{0i})} \boldsymbol{\delta}_{\xi,\xi_{0i}}$$
$$= -\iiint_{-\infty}^{\infty} d\xi' \, d\eta' \, d\zeta' \, \phi(\xi', \eta', \zeta') \int_B \frac{\partial}{\partial \nu} F(\xi' - \xi, \eta' - \eta, \zeta' - \zeta)\tau(\xi, \eta, \zeta) \, dS,$$

Here $\int_B dS$ denotes integration over the surfaces formed by the trace of $(\xi_{0i}, \eta_{0i}, \zeta_{0i})$, and $\partial/\partial\nu$ is an inward (towards the region with $H = 1$) normal derivative at dS. Moreover, this integration should always be understood in the sense of Hadamard's finite integral as well as Cauchy's principal value; indeed if F is singular at a certain point, we have the additional term in (A19) on account of (A17) with $\Delta F_0 = \infty$, so that it may cancel out the infinity caused by a highly singular kernel in the integral term. (This may be seen by investigating a simplified equation based on (A17).) Thus, all integrations over the wing and wake in the text are understood in the same sense. If F has a finite discontinuity, obviously the corresponding additional term derived from (A17) should be added on the right-hand side of (A19).

References

1. G. N. Ward, *Linearized theory of steady high-speed flow,* Cambridge University Press, 1955
2. Th. von Kármán, *The similarity law of transonic flow,* J. Math. Phy. **26** (1947) 182

3. K. Oswatitsch and F. Keune, *The flow around bodies of revolution at Mach number 1*, Proc. Conf. on High-Speed Aeronautics, Polytech. Inst. Brooklyn, N. Y., 1955, p. 113
4. P. F. Maeder and H. U. Thommen, *Some results of linearized transonic flow about slender airfoils and bodies of revolution*, J. Aero. Sci. **23** (1956) 187
5. I. Hosokawa, *A refinement of the linearized transonic flow theory*, J. Phys. Soc. Japan **15**, (1960) 149
6. P. F. Maeder and H. U. Thommen, *Linearized transonic flow about slender bodies at zero angle of attack*, J. Appl. Mech. **28** (1961) 481
7. J. R. Spreiter and A. Y. Alksne, *Thin airfoil theory based on approximate solution of the transonic flow equation*, NACA Rep. 1359 (1958), and *Slender body theory based on approximate solution of the transonic flow equation*, NASA Rep. 2 (1959)
8. K. R. Dorfner, *Dreidimensionale Überschallprobleme der Gasdynamik*, Springer-Verlag, 1957
9. L. Schwartz, *Théorie des distributions*, Hermann, Paris, Vol. I (1950), Vol. II (1951)
10. J. R. Spreiter, *On alternative forms for the basic equations of transonic flow theory*, J. Aero. Sci. **20**, (1954) 70
11. A. Robinson and J. A. Laurmann, *Wing theory*, Cambridge University Press, 1956, p. 225
12. N. I. Muskhelishvili, *Singular integral equations*, Noordhoff, Groningen, 1953 (translation of the 2nd ed., Moscow, 1946)