$\psi(x) = \beta(a + bx^2) + 0(h^{-5}), \phi(x) = -\beta(cx + dx^3) + 0(h^{-5})$, where

$$a = 1 + \frac{A_0}{h} + \frac{A_0^2}{h^2} + \frac{6A_0^3 - A_1}{6h^3} + \frac{6A_0^4 - 3A_0A_1 - 2B_0^2}{6h^4},$$

$$b = -\frac{A_1}{2h^3} - \frac{A_0A_1}{2h^4}, \qquad c = \frac{B_0}{h^2} + \frac{A_0B_0}{h^2} + \frac{6A_0^2B_0 - B_1}{6h^4}, \qquad d = -\frac{B_1}{6h^4}.$$

It is then a simple matter to obtain

$$u_z(r, h) = (2\epsilon/\pi)(1 - r^2)^{1/2}[c + d(4r^2 - 1)/3] + 0(h^{-5}), \qquad 0 \le r \le 1,$$

$$\sigma_{zz}(r, 0) = -\beta(1 - r^2)^{-1/2}(a - b + 2br^2) + 0(h^{-5}), \qquad 0 \le r < 1.$$

It is interesting to note from the above expression for $u_z(r, h)$ that since $c = 0(h^{-2})$, the effect of the hole in the foundation is negligible when $h$ is so large that terms of order $h^{-2}$ can be ignored. For in this case $u_z(r, h)$ would be zero for all $r$, which is the boundary condition one would use in place of (4) and (6) when the foundation contains no hole.

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# ON THE OSCILLATIONS OF A PENDULUM UNDER PARAMETRIC EXCITATION*

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In a recent paper [1], we examined the oscillations of a pendulum under parametric excitation using a formal asymptotic method. It is the purpose of this note to point out that the general behavior suggested for this system may be corroborated through the application of a new mapping theorem due to Moser [2].

Consider the nonlinear equation

$$\frac{d^2\theta}{dt^2} + \left(\omega_0^2 - \frac{\xi_0\omega^2}{L}\cos\omega t\right)\sin\theta = 0, \tag{1}$$

where $\omega$, $\omega_0$, $\xi_0$ and $L$ are positive constants with $\xi_0/L$ small. This equation depicts the motion of a simple pendulum which is excited parametrically by small, vertical vibrations of its support. (See [1]). The free motions for $\xi_0 = 0$ are well known [3] and can be described by the energy integral

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta = E. \tag{2}$$

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There are both oscillatory and rotary motions which are periodic and depend upon the magnitude of the energy constant $E$. The first result we wish to state concerns nonresonant solutions of Eq. (1) in a neighborhood of one of these periodic free motions in the appropriate phase plane. (For oscillatory motion, $\theta$ and $d\theta/dt$ become rectangular coordinates while for rotary motion, $\theta$ and $d\theta/dt$ become polar coordinates.)

*Theorem* 1 (Moser): Let $\Gamma_E$ denote a periodic trajectory (2) in the phase plane and let $P_E$ denote the corresponding fundamental period. Then there exists a positive $\delta$, depending upon $\Gamma_E$ and upon a positive number $\rho$ (See (3)), such that for all $\xi_0/L < \delta$ and nearly all excitation frequencies there exists a manifold of almost periodic solutions of (1) with basic frequencies $2\pi/P_E$ and $\omega$ and which emanate from all points of a closed curve $\Gamma$ lying near $\Gamma_E$. Specifically, it is required that $\omega$ satisfy the inequality

$$\left| m\omega - \frac{2n\pi}{P_E} \right| \geq \rho n^{-3/2} \tag{3}$$

for all integers $m$ and positive integers $n$. (The numbers $\omega$ so excluded have a density which is small if the number $\rho$ is small.) A proof of this theorem is given in [4].

The frequencies which violate (3) are the resonant frequencies of (1), i.e. the frequencies which are nearly rational multiples of the natural frequency, and necessitate special treatment. As the excitation frequency $\omega$ approaches one of these resonant frequencies one might expect to observe a long-period beat superimposed upon an otherwise periodic response, as suggested in [1]. For an oscillatory type motion of small amplitude, this expectation is borne out and is exemplified by Theorem 2 below.

Following [1], let us introduce the dimensionless time $\tau = \omega_0 t$ and put

$$\epsilon = \xi_0/L, \qquad \theta = \epsilon^{1/2} X, \qquad \eta = \omega/\omega_0\,, \qquad m = (2 - \eta)/\epsilon.$$

Then, if we define new dependent variables $(a, \gamma)$ through the transformation

$$X = a \cos \tfrac{1}{2}(\eta\tau - \gamma), \qquad \frac{dX}{d\tau} = -\frac{\eta}{2}\, a \sin \tfrac{1}{2}(\eta\tau - \gamma), \tag{4}$$

Equation (1) can be shown to be equivalent to a system of the form

$$\frac{1}{\epsilon}\frac{d\gamma}{d\tau} = -m + \frac{a^2}{8} + \frac{\eta^2}{2}\cos\gamma + 2\eta\cos\eta\tau + \left(\frac{\eta a^2}{12} - m\right)\cos(\eta\tau - \gamma)$$
$$+ \eta\cos(2\eta\tau - \gamma) + \frac{\eta a^2}{48}\cos(2\eta\tau - 2\gamma) + \epsilon f_1(\epsilon, a, \gamma, \tau),$$

$$\frac{1}{\epsilon}\frac{da}{d\tau} = \frac{\eta^2 a}{4}\sin\gamma - \left(\frac{\eta a^3}{48} - \frac{m}{2}\,\mathrm{a}\right)\sin(\eta\tau - \gamma) - \frac{\eta a}{2}\sin(2\eta\tau - \gamma)$$
$$- \frac{\eta a^3}{96}\sin(2\eta\tau - 2\gamma) + \epsilon f_2(\epsilon, a, \gamma, \tau), \tag{5}$$

where the detuning parameter $m$ is to be held constant as $\epsilon \to 0$. This is the primary resonance case in which the frequency ratio $\eta \to 2$. The higher order terms $f_1$ and $f_2$ in (5) are analytic in $(a, \gamma)$ and are periodic in $\tau$ of period $2\pi/\eta$.

A first approximation to this system (which may be obtained here by averaging with respect to $\tau$ the first order terms in (5)) has been discussed in [1]. The approximate solutions are described by an energy type integral

$$-ma^2 + \frac{a^4}{16} + \frac{\eta^2 a^2}{2}\cos\gamma = c \tag{6}$$

which represents one or more families of closed trajectories in the phase plane (here, $a$ and $\gamma$ become polar coordinates) with appropriate separatrices and singular points. The closed trajectories depict periodic solutions (long-period beats of the motion) and the singular points correspond to stationary oscillations, i.e. periodic solutions of (1). By arguments similar to those used in [5] one encounters no difficulty in proving the following:

*Theorem* 2: Let $\Gamma_c$ denote a closed trajectory (6) of the averaged system. Then there is a positive $\epsilon_0$ , depending upon $\Gamma_c$ and upon a positive number $\rho$ (see (7)), such that for nearly all $\epsilon$ with $|\epsilon| < \epsilon_0$ there exists a manifold of almost periodic solutions of (5) which emanate from all points of a closed curve $\Gamma$ lying near $\Gamma_c$ . The excluded values of $\epsilon$ have a density near $\epsilon = 0$ which is small if the number $\rho$ is small. Specifically, it is required that $\epsilon$ satisfy the auxiliary inequality

$$\left| mT_c - \frac{2m\pi\epsilon}{\eta} \right| \geq \rho n^{-3/2} \tag{7}$$

for all integers $m$ and positive integers $n$, where $T_c$ is the fundamental period in slow time $s = \epsilon\tau$ associated with $\Gamma_c$ . (See [1], where $T_c = \epsilon\Delta\tau$.) The basic frequencies in $\tau$ of these almost periodic solutions of (5) are $2\pi\epsilon/T_c$ and $\eta$. The former represents the fundamental beat frequency while the latter merely corresponds to an overtone of the primary oscillation in (4).

The excluded values of $\epsilon$ in Theorem 2 are those for which the beat frequency and the primary oscillation frequency $\eta/2$ are nearly resonant for a given (relative) detuning $m$. These correspond to second order resonance cases and could, conceivably, lead to beats with frequencies of the order of $\epsilon^2$. In any event, the numerous manifolds of almost periodic solutions of (1) described by Theorems 1 and 2 for various choices of generating orbits (2) and (6) can be expected to be distributed throughout the pertinent regions of the solution spaces rather completely and will, therefore, restrict other (not necessarily almost periodic) solutions to small neighborhoods of these orbits.

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