ON THE DAMPING OF A SATELLITE MOTION*

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Abstract. The motion in a circular orbit of a gravitationally oriented satellite, whose angular motions about the local vertical are damped by a roll-vee, gyrostabilizer system, is considered, wherein the pitch axis of the satellite remains perpendicular to the orbital plane. It is shown that no matter how large the initial local angular velocity of the satellite is, this velocity reaches any given smaller value in a finite time. It is also shown how it is possible to obtain a bound on this damping time.

1. Introduction and summary. A description of an attitude control system for a gravitationally oriented earth-pointing satellite has been given by J. A. Lewis and E. E. Zajac [1]. The attitude control system involves two single-degree-of-freedom gyroscopes in a so-called roll-vee configuration. In the desired motion, the satellite describes a circular orbit and has one axis continuously aligned along the local vertical. If at any given instant the satellite is not aligned along the local vertical with the proper angular velocity of one rotation per orbit, the gyroscopes precess about their output axes and undergo viscous damping, so that at least for small initial disturbances the satellite finally settles out in one of two possible earth-pointing positions.

It is of importance to be sure that the gyroscopes will damp out the angular librations of the satellite for arbitrary initial conditions, including the case of tumbling. Heretofore the only available evidence has consisted of plausibility arguments and a limited number of numerical solutions of the differential equations of motion, for moderately large initial angular rates. The present paper contains a rigorous proof that for a limited class of motions (that is, pure pitching motions), satellite damping occurs for arbitrarily large initial angular rates.

In the motions considered, the satellite revolves in a circular orbit and rotates so that its so-called "pitch" axis remains perpendicular to the orbital plane. The differential equations describing this motion take the form

$$\beta^{\cdot\cdot} + a^2 \sin 2\beta + p\varphi^{\cdot} \sin (\alpha - \varphi) = 0, \tag{1.1}$$

$$\varphi^{\cdot} + \mu q \varphi + q[\sin \alpha - (1 + \beta^{\cdot}) \sin (\alpha - \varphi)] = 0, \tag{1.2}$$

with the initial conditions

$$\varphi(0) = \varphi_0 , \qquad -\left(\frac{\pi}{2} - \alpha\right) < \varphi_0 < \alpha; \qquad \beta(0) = 0; \qquad \beta^{\cdot}(0) = \omega_0 > 0, \tag{1.3}$$

where β is the angle of rotation of the satellite measured with respect to the local vertical, and the dot superscript denotes differentiation with respect to time. The gyros are symmetrically situated with respect to the pitch axis and hence only one gimbal angle φ appears above. It is assumed that the gyro stops are at $\varphi = \alpha$ and $\varphi = \alpha - (\pi/2)$, i.e., there are ideal stops along the pitch and yaw axes. The constants satisfy

$$a > 0; \qquad \mu \geq 0; \qquad p > 0; \qquad q > 0; \qquad 0 < \alpha < \frac{\pi}{2}. \tag{1.4}$$

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The question of interest here is whether or not it is possible for the satellite to undergo such motion without damping occurring. We show in the next section that, given any Ω with $0 < \Omega < \omega_0$, $\beta^{\cdot}$ cannot be greater than Ω for all $t > 0$. The proof is by contradiction. A numerical investigation has been made[1] of the damping of the satellite for moderate values of ω_0. The result here shows that, no matter how large the initial local angular velocity of the satellite is, this velocity reaches any given smaller value within a finite time. In the final section we show how it is possible to obtain a bound on the time taken for $\beta^{\cdot}$ to first reach the given value Ω.

2. Proof by contradiction. We will assume that $\beta^{\cdot} > \Omega$ for $t > 0$, where $0 < \Omega < \omega_0$, and obtain a contradiction. Under the assumption we may change from the independent variable t to the variable β. Thus, let

$$\beta^{\cdot} = F(\beta); \qquad \psi = (\alpha - \varphi). \tag{2.1}$$

Then, from Eqs. (1.1) to (1.3), using primes to denote β derivatives,

$$FF' + a^2 \sin 2\beta = pF\psi' \sin \psi, \tag{2.2}$$

$$F\psi' = q[\sin \alpha - (1 + F) \sin \psi] + \mu q(\alpha - \psi), \tag{2.3}$$

with the initial conditions

$$F(0) = \omega_0 ; \qquad \psi(0) = \psi_0 = (\alpha - \varphi_0), \qquad 0 < \psi_0 < \frac{\pi}{2}. \tag{2.4}$$

Since, by assumption, $F > \Omega > 0$ it follows from Eqs. (2.3) and (1.4) that

$$0 < \psi \leq \max(\alpha, \psi_0) < \frac{\pi}{2}. \tag{2.5}$$

Now, from Eqs. (2.2) and (2.3) we may obtain the energy integral

$$F^2 + a^2(1 - \cos 2\beta) + 2p \int_{\psi}^{\alpha} (\sin \alpha - \sin \theta)\, d\theta + p\mu(\alpha - \psi)^2 + \frac{2p}{q} \int_0^{\beta} F\psi'^2\, d\beta$$

$$= p\mu\varphi_0^2 + 2p \int_{\psi_0}^{\alpha} (\sin \alpha - \sin \theta)\, d\theta + \omega_0^2 \equiv \omega^2, \tag{2.6}$$

using the initial conditions in Eq. (2.4). From Eqs. (2.5) and (2.6) it follows that

$$0 < \Omega < F \leq \omega, \tag{2.7}$$

and that

$$\int_0^{\beta} \psi'^2\, d\beta \leq \frac{1}{\Omega} \int_0^{\beta} F\psi'^2\, d\beta \leq \frac{q(\omega^2 - \Omega^2)}{2p\Omega}, \tag{2.8}$$

for all $\beta > 0$. Hence, if

$$I_n = \int_{n\pi}^{(n+1)\pi} \psi'^2\, d\beta, \tag{2.9}$$

then $I_n \to 0$ as $n \to \infty$.

We will suppose that

$$|\delta| \leq \frac{\pi}{2}, \tag{2.10}$$

and let

$$\gamma_n = \psi\left(n\pi + \frac{\pi}{2}\right). \tag{2.11}$$

Then, using Schwarz's inequality, we obtain

$$\left| \psi\left(n\pi + \frac{\pi}{2} + \delta\right) - \gamma_n \right| = \left| \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} \psi' \, d\beta \right|$$
$$\leq \left[\delta \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} \psi'^2 \, d\beta \right]^{1/2} \leq \left(\frac{\pi}{2} I_n\right)^{1/2}, \tag{2.12}$$

from Eqs. (2.9) and (2.10). Now, from Eq. (2.2),

$$[F^2(\beta) - a^2 \cos 2\beta]' = 2pF\psi' \sin \psi. \tag{2.13}$$

Hence,

$$F^2\left(n\pi + \frac{\pi}{2} + \delta\right) = \lambda_n^2 + a^2(1 - \cos 2\delta) + \nu_n(\delta), \tag{2.14}$$

where

$$\lambda_n = F\left(n\pi + \frac{\pi}{2}\right); \qquad \nu_n(\delta) = 2p \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} F\psi' \sin \psi \, d\beta. \tag{2.15}$$

From Eqs. (2.5), (2.7), (2.9) and (2.10) it follows that

$$| \nu_n(\delta) | \leq 2p\omega \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} | \psi' | \, | d\beta | \leq 2p\omega\left(\frac{\pi}{2} I_n\right)^{1/2}, \tag{2.16}$$

again using Schwarz's inequality.

Finally, from Eqs. (2.3) and (2.11),

$$\left[\psi\left(n\pi + \frac{\pi}{2} + \delta\right) - \gamma_n\right] = q \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} \{[(\sin \alpha - \sin \psi)$$
$$+ \mu(\alpha - \psi)]/F - \sin \psi\} \, d\beta. \tag{2.17}$$

Thus,

$$\left[\psi\left(n\pi + \frac{\pi}{2} + \delta\right) - \gamma_n\right] + q \int_{(2n+1)\pi/2}^{\delta+(2n+1)\pi/2} \left\{\frac{[(\sin \alpha - \sin \gamma_n) + \mu(\alpha - \gamma_n)]}{[\lambda_n^2 + a^2(1 + \cos 2\beta)]^{1/2}}\right.$$
$$\left. - [(\sin \alpha - \sin \psi) + \mu(\alpha - \psi)]/F + (\sin \psi - \sin \gamma_n)\right\} d\beta$$
$$= q \int_0^\delta \left\{\frac{[(\sin \alpha - \sin \gamma_n) + \mu(\alpha - \gamma_n)]}{[\lambda_n^2 + a^2(1 - \cos 2\theta)]^{1/2}} - \sin \gamma_n\right\} d\theta. \tag{2.18}$$

But, from Eqs. (2.5), (2.7), (2.11) and (2.15),

$$0 < \gamma_n < \max(\alpha, \psi_0) < \frac{\pi}{2}; \qquad 0 < \Omega < \lambda_n \leq \omega. \tag{2.19}$$

Since $I_n \to 0$ as $n \to \infty$ it follows, from Eqs. (2.12), (2.14) and (2.16), that $| \psi(n\pi + \pi/2 + \delta) - \gamma_n | \to 0$ and $| F^2(n\pi + \pi/2 + \delta) - \lambda_n^2 - a^2(1 - \cos 2\delta) | \to 0$, uniformly in $|\delta| \leq \pi/2$, as $n \to \infty$. Thus the left-hand side of Eq. (2.18) tends to zero, uniformly in $|\delta| \leq \pi/2$, as $n \to \infty$. But, since $[(\sin \alpha - \sin \gamma_n) + \mu(\alpha - \gamma_n)]$ and $\sin \gamma_n$ cannot simultaneously tend to zero, and $[\lambda_n^2 + a^2(1 - \cos 2\theta)]^{-1/2}$ is uniformly bounded away from zero, and is not constant, the right-hand side of Eq. (2.18) cannot tend to zero, for all $|\delta| \leq \pi/2$, as $n \to \infty$. Thus we are led to a contradiction. Consequently, whatever the value of Ω, with $0 < \Omega < \omega_0$, the quantity F, i.e., $\beta^{\cdot}$, attains the value Ω within a finite time.

3. Bound on a damping time. We may use a similar approach to obtain an upper bound on the time taken for $\beta^{\cdot}$ to first reach a given value Ω, with $0 < \Omega < \omega_0$. Thus, suppose that $\beta^{\cdot} = F > \Omega$ for $0 \leq \beta \leq m\pi$, for some positive integer m, and consider this range of β. Then, from Eqs. (2.8) and (2.9),

$$\frac{q(\omega^2 - \Omega^2)}{2p\Omega} \geq \int_0^{m\pi} \psi'^2 \, d\beta = \sum_{n=0}^{(m-1)} I_n . \tag{3.1}$$

If we show that

$$I_n \geq I > 0, \tag{3.2}$$

then it follows that

$$m \leq \frac{q(\omega^2 - \Omega^2)}{2p\Omega I}. \tag{3.3}$$

Hence, F must reach the value Ω for some $\beta = \beta^*$, where

$$\beta^* < \pi\left\{1 + \left[\frac{q(\omega^2 - \Omega^2)}{2p\Omega I}\right]\right\}. \tag{3.4}$$

Since, from Eq. (2.1),

$$t = \int_0^{\beta} \frac{d\gamma}{F(\gamma)}, \tag{3.5}$$

the corresponding time is

$$T \leq \beta^*/\Omega. \tag{3.6}$$

It remains to determine I in Eq. (3.2).

Now, from Eqs. (2.3) and (2.9),

$$I_n = q^2 \int_{n\pi}^{(n+1)\pi} \{[(\sin \alpha - \sin \psi) + \mu(\alpha - \psi)]/F - \sin \psi\}^2 \, d\beta. \tag{3.7}$$

Let

$$K_n = q^2 \int_{n\pi}^{(n+1)\pi} \left\{\frac{[(\sin \alpha - \sin \gamma_n) + \mu(\alpha - \gamma_n)]}{[\lambda_n^2 + a^2(1 + \cos 2\beta)]^{1/2}} - \sin \gamma_n\right\}^2 d\beta. \tag{3.8}$$

Define

$$S(\beta) \equiv \{[(\sin \alpha - \sin \psi) + \mu(\alpha - \psi)]/F - \sin \psi\} + \left\{\frac{[(\sin \alpha - \sin \gamma_n) + \mu(\alpha - \gamma_n)]}{[\lambda_n^2 + a^2(1 + \cos 2\beta)]^{1/2}} - \sin \gamma_n\right\}. \tag{3.9}$$

Now,

$$|\sin x - \sin y| = 2\left|\sin\left(\frac{x - y}{2}\right)\cos\left(\frac{x + y}{2}\right)\right| \leq 2 \sin \left|\frac{x - y}{2}\right| \leq |x - y|. \tag{3.10}$$

Also, from Eqs. (2.4), (2.5) and (2.19),

$$|\alpha - \psi| \leq \max(\alpha, |\varphi_0|); \qquad |\alpha - \gamma_n| \leq \max(\alpha, |\varphi_0|). \tag{3.11}$$

Hence, using Eqs. (2.7) and (2.19),

$$|S(\beta)| \leq 2\left[1 + \frac{(1+\mu)}{\Omega}\max(\alpha, |\varphi_0|)\right] \equiv A. \tag{3.12}$$

We next define

$$\begin{aligned} D(\beta) \equiv &\{[(\sin\alpha - \sin\psi) + \mu(\alpha - \psi)]/F - \sin\psi\} \\ &- \left\{\frac{[(\sin\alpha - \sin\gamma_n) + \mu(\alpha - \gamma_n)]}{[\lambda_n^2 + a^2(1 + \cos 2\beta)]^{1/2}} - \sin\gamma_n\right\} \\ &= (\sin\gamma_n - \sin\psi) + [(\sin\gamma_n - \sin\psi) + \mu(\gamma_n - \psi)]/F \\ &+ \frac{[(\sin\alpha - \sin\gamma_n) + \mu(\alpha - \gamma_n)][\lambda_n^2 + a^2(1 + \cos 2\beta) - F^2]}{F[\lambda_n^2 + a^2(1+\cos 2\beta)]^{1/2}\{F + [\lambda_n^2 + a^2(1 + \cos 2\beta)]^{1/2}\}}. \end{aligned} \tag{3.13}$$

From Eqs. (2.12), (2.14) and (2.16),

$$|\psi - \gamma_n| \leq \left(\frac{\pi}{2} I_n\right)^{1/2}; \qquad |F^2 - \lambda_n^2 - a^2(1 + \cos 2\beta)| \leq 2p\omega\left(\frac{\pi}{2} I_n\right)^{1/2}, \tag{3.14}$$

for $n\pi \leq \beta \leq (n+1)\pi$. Hence, using Eqs. (2.7), (2.19), (3.10) and (3.11), we have

$$|D(\beta)| \leq \left(\frac{\pi}{2} I_n\right)^{1/2}\left[1 + \frac{(1+\mu)}{\Omega} + \frac{p\omega}{\Omega^3}(1+\mu)\max(\alpha, |\varphi_0|)\right] \equiv B\left(\frac{\pi}{2} I_n\right)^{1/2}, \tag{3.15}$$

for $n\pi \leq \beta \leq (n+1)\pi$.

From Eqs. (3.7), (3.8), (3.9), (3.12), (3.13) and (3.15), it follows that

$$\begin{aligned} |I_n - K_n| &= q^2 \left| \int_{n\pi}^{(n+1)\pi} S(\beta) D(\beta)\, d\beta \right| \\ &\leq q^2 \pi A B \left(\frac{\pi}{2} I_n\right)^{1/2} \equiv (2CI_n)^{1/2}, \end{aligned} \tag{3.16}$$

where C is a constant independent of n. From Eq. (3.16) we deduce that

$$I_n \geq [(C + K_n) - \{C(C + 2K_n)\}^{1/2}]. \tag{3.17}$$

Also, from Eqs. (2.19) and (3.8), it follows that

$$K_n \geq K > 0, \tag{3.18}$$

where K is independent of n. Finally, from Eqs. (3.17) and (3.18),

$$I_n \geq [(C + K) - \{C(C + 2K)\}^{1/2}] \equiv I. \tag{3.19}$$

Undoubtedly the bounds in Eqs. (3.4) and (3.6) are rather poor in general. Improvements are possible, but it does not seem worthwhile to detail them here.

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Reference

1. J. A. Lewis and E. E. Zajac, *The roll-vee, two-gyro satellite attitude control system,* to appear in the Bell System Technical Journal.