THE SPECTRA OF IRROTATIONAL FLOWS*

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1. Introduction. In the n-dimensional Euclidean space of points $x = (x_1, x_2, \cdots, x_n)$, let $f(x)$ denote a vector of class C^1 on some region (connected open set) R. Suppose that R is an invariant set of the system of equations

$$x' = f(x) \qquad (x' = dx/dt), \tag{1}$$

so that if x_0 is any point of R, then the solution $x = x(t)$ of (1), satisfying $x(0) = x_0$, exists and lies in R, for $-\infty < t < \infty$. It will be supposed that there exists a positive function $\rho = \rho(x)$ of class C^1 in R for which

$$\operatorname{div}(\rho f) = 0. \tag{2}$$

Then the system (1) has an invariant measure m, where $dm = \rho\, dx$, on R. Thus, $m(T_t(A)) = m(A)$ if A is a measurable set in R and T_t denotes the transformation

$$T_t : p = x(0) \to p_t = x(t) \tag{3}$$

determined by (1). See Hopf [2], p. 8.

The transformation $g(p) \to g(p_t)$, together with the measure m, determines a unitary transformation U_t of the Hilbert space $L^2(R)$ into itself. For each fixed t, let $U = U_t$ have the spectral resolution

$$U = \int_0^{2\pi} e^{i\lambda}\, dE(\lambda) \qquad (E(\lambda) = E_t(\lambda)). \tag{4}$$

The operator U will be called absolutely continuous if $(E(\lambda)g, h)$ is an absolutely continuous function of λ for each pair of functions g, h of $L^2(R)$. In case $t = 0$, then $U = I$, the identity transformation.

The zeros (in R) of the vector f constitute the set E of equilibrium points of (1). Thus, if $p = x(0)$ is in E, the solution $x(t)$ of (1) satisfies $x(t) = x(0)$ for $-\infty < t < \infty$. It is clear that E is an invariant set and that the restriction of U_t to E is, for all t, the identity transformation. Moreover, since f is continuous, it is clear that the intersection of E with any closed subset of R is a closed set. It follows that $E \neq R$ if and only if $m(R - E) > 0$.

There will be proved the following

THEOREM. *Let* (1) *satisfy* (2) *and in addition let there exist a function* $\phi = \phi(x)$ *of class* C^2 *on* R *for which*

$$f = \operatorname{grad} \phi. \tag{5}$$

(i) *If* $E \neq R$, *then the restriction of* U_t *to* $R - E$ *is for all* $t \neq 0$ *an absolutely continuous unitary operator whose spectrum is the entire unit circle* $|z| = 1$. (ii) *If, in addition to* (5), *the system* (1) *satisfies, instead of* (2), *the incompressibility condition*

$$\operatorname{div} f = 0, \tag{2'}$$

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so that dm can be taken to be dx, then either $R = E$ *(hence* $U_t = I$ *on R for* $-\infty < t < \infty$*) or E is a zero set. In the latter case, by* (i), U_t *is for* $t \neq 0$ *absolutely continuous on R and has the unit circle as its spectrum.*

2. Remarks. Conditions (2) and (5) imply that $m(R - E) = \infty$ whenever $E \neq R$. For (ii) asserts that U_t for $t \neq 0$ is absolutely continuous on $R - E$ and hence, in particular, U_t has no point spectrum. But, if $0 < m(R - E) < \infty$, then the characteristic function of $R - E$ would be an eigenfunction of U_t belonging to the eigenvalue 1, a contradiction.

In case (2′) and (5) hold, so that (1) is incompressible and irrotational, then by (ii), either $R = E$ or E is a zero set and so, as noted above, $m(R) = \infty$. It is interesting to compare this result with an exercise in Kellogg [3], p. 215, Ex. 2. It can be noted that when $n = 3$ and R is simply connected the condition (5) is a consequence of the irrotationality assumption on the flow; cf., e.g., Kellogg [3], p. 74.

Special cases of the theorem were considered in [4], cf., p. 229. When $n = 1$, the system $x' = 1$ is incompressible and has solutions $x(t) = t + x(0)$; the unitary operator U_t is then the translation operator on $-\infty < t < \infty$. It follows from the above Theorem (and was also shown in [4]) that U_t is absolutely continuous with the spectrum $|z| = 1$. This last assertion can also be deduced directly from considerations similar to those given in Hille [1], pp. 329 ff.

3. Proof of the Theorem. In order to prove (i), suppose that $E \neq R$ and that $t > 0$. Then for p in R and t fixed, it follows from (1) and (5) that $d\phi/du = |\operatorname{grad} \phi|^2$ along the path p_u from $u = 0$ to $u = t$ ($p = p_0$) and hence

$$d(p) = \phi(p_t) - \phi(p) = \int_0^t |\operatorname{grad} \phi|^2 \, du. \tag{6}$$

If p is in $R - E$ then clearly $d(p) > 0$. The desired assertion (i) now follows from the theorem of [4], p. 228.

In order to prove (ii), note that relations (2′) and (5) imply that ϕ is harmonic in R. If ψ denotes any one of the components of grad ϕ then ψ is also harmonic and $\psi = 0$ on E. Hence if E has positive measure, then $\psi \equiv 0$ on R,* and so $R = E$.

REFERENCES

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2. E. Hopf, *Ergodentheorie*, Chelsea Publishing Co., New York, 1948
3. O. D. Kellogg, *Foundations of Potential Theory*, Springer, Berlin, 1929
4. C. R. Putnam, *Commutators, perturbations and unitary spectra*, Acta Math., **106** (1961) 215–232

*What is needed here is the following fact: If C is a closed set of positive measure contained in some region R and if ψ is harmonic in R and satisfies $\psi = 0$ on C, then $\psi \equiv 0$ on R. Although this is well-known in case C contains an open set (at least if $n \leqq 3$), the author did not find a reference in the literature dealing explicitly with the problem at hand. A proof can be obtained by noting that ψ is a real analytic function of n variables in a region R and that, unless $\psi \equiv 0$ in R, the zeros of such a function form a set of n-dimensional Lebesgue measure zero. A proof of this last fact using the Lebesgue density theorem was pointed out to the author by H. Flanders. Another proof can be obtained by noting, first, that for $n = 1$ the assertion in question is a consequence of the identity theorem for power series in one variable and, second, that for a power series in n variables, it can be deduced from the corresponding assertion in the one dimensional case by an application of Fubini's theorem for multiple integrals.