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ON THE SPECTRA OF UNITARY HALF-SCATTERING OPERATORS*

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1. Introduction. Let H_0 denote a self-adjoint operator, bounded or unbounded, on a Hilbert space of elements f, g, $\cdots$, and suppose that H_0 is absolutely continuous. (By the absolute continuity of a self-adjoint operator H with the spectral resolution

$$H = \int_{-\infty}^{\infty} \lambda \, dE(\lambda) \tag{1}$$

is meant that $\| E(\lambda)f \|$ is an absolutely continuous function of λ for every element f; see Rosenblum [15], Kato [10], Kuroda [12], Putnam [13].) Let V denote a self-adjoint operator for which H_1 , where

$$H_1 = H_0 + V \tag{2}$$

is self-adjoint and absolutely continuous. The operator H_1 defined by (2) must be self-adjoint if, for instance, V is bounded; non-trivial conditions assuring the absolute continuity of H_1 however are not so apparent.

In [3] and [4], Friedrichs considered a special pair H_0 and V, where H_0 was bounded (and also absolutely continuous) and V was a certain type of integral operator, and showed that if H_1 is defined by

$$H_1^\epsilon = H_0 + \epsilon V, \qquad \epsilon \text{ real}, \tag{3}$$

then, for sufficiently small ϵ, the strong limits

$$\lim_{t \to \infty} U_t^\epsilon = U_+^\epsilon \quad \text{and} \quad \lim_{t \to -\infty} U_t^\epsilon = U_-^\epsilon , \quad \text{where} \quad U_t^\epsilon = \exp(itH_1^\epsilon) \exp(-itH_0), \tag{4}$$

exist, and U_+^ϵ and U_-^ϵ are unitary operators satisfying

$$H_1^\epsilon = U_+^\epsilon H_0 U_+^{\epsilon *} \quad \text{and} \quad H_1^\epsilon = U_-^\epsilon H_0 U_-^{\epsilon *}. \tag{5}$$

In addition, each of the operators $U^\epsilon = U_+^\epsilon$, U_-^ϵ is analytic in ϵ,

$$U^\epsilon = I + \epsilon U_1 + \epsilon^2 U_2 + \cdots , \tag{6}$$

and hence satisfies

$$\| U^\epsilon - I \| \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{7}$$

It is seen that condition (7) implies that

$$sp(U^\epsilon) \to 1 \quad \text{as} \quad \epsilon \to 0, \tag{8}$$

where sp(A) denotes the spectrum of an operator A.

Later, Rosenblum [15] and Kato [9], [10], showed that if (2) holds, where H_0 and H_1 are absolutely continuous and if V is of the trace class (that is, V is completely continuous with eigenvalues λ_1, λ_2, $\cdots$ satisfying $\sum |\lambda_i| < \infty$) then the strong limits

$$U_+ = \lim_{t \to \infty} U_t \quad \text{and} \quad U_- = \lim_{t \to -\infty} U_t, \quad \text{where} \quad U_t = \exp(itH_1)\exp(-itH_0), \qquad (9)$$

exist and are unitary operators satisfying

$$H_1 = U_+ H_0 U_+^* \quad \text{and} \quad H_1 = U_- H_0 U_-^*. \qquad (10)$$

Similar problems, including that of the perturbation of absolutely continuous spectra, as well as generalizations to cases where H_0 and H_1 are not necessarily absolutely continuous, where V is not necessarily of the trace class, and where the limits (9), when they exist, may be only partially isometric, have been investigated. In particular, see Aronszajn [1], Kato [9], [10], Kuroda [12], Rosenblum [15], and, especially in the physical context of scattering theory, Cook [2], Hack [6], Jauch [7], Jauch and Zinnes [8]. (In the physical theory, the operators H_0, H_1 and V correspond to the free Hamiltonian, total Hamiltonian and interaction potential respectively, see [7] and [8]; the operators U_+ and U_- of (9) have been called half-scattering operators by Friedrichs [5], p. 233, and wave operators by Jauch [7], p. 137.) When (3) is assumed and H_0 and V satisfy certain conditions, these results imply the existence of the operators U_+^ϵ and U_-^ϵ, as strong limits defined by (4) and satisfying (5). However, $U^\epsilon = U_+^\epsilon$, U_-^ϵ now *may* not be analytic perturbations of the identity, (6), and so (7) and (8) *may* not hold. It will be shown in the present paper that under certain conditions relation (8), hence also (7) and (6), *must* not hold.

Let H_0 and H_1 denote the free and perturbed Hamiltonian one-dimensional wave mechanical operators

$$H_0 = -d^2/dx^2, \qquad H_1 = H_0 + V(x), \qquad (11)$$

on the Hilbert space $L^2(-\infty, \infty)$. It will be shown, as a consequence of a result of Kuroda [12], that if $V = V(x)$ is non-negative, bounded, and "small" for large x (condition (15)) then necessarily the strong limits of (9) exist as unitary operators, and that furthermore, as a consequence of [14], if also $V(x)$ is "not too small" (condition (16)), then the spectra of these operators are the entire unit circle $|z| = 1$. Correspondingly, if H_1^ϵ is defined by (3), then (4) and (5) hold, and $sp(U_+^\epsilon)$ and $sp(U_-^\epsilon)$ are, for each $\epsilon > 0$, also the unit circle. In this case then, relation (8) fails to hold, and U_+^ϵ and U_-^ϵ surely cannot be of the form (6). Use will be made of the following

Lemma. On a Hilbert space let H_0 be a non-negative (not necessarily bounded) self-adjoint operator, V a bounded non-negative self-adjoint operator, and suppose that H_0 and $H_1 = H_0 + V$ are unitarily equivalent, thus

$$H_0 \geqq 0, \qquad 0 \leqq V \leqq \text{const.} \, I, \qquad H_1 = UH_0U^* \qquad (U, \text{ unitary}). \qquad (12)$$

If f is any element of the Hilbert space for which $g = V^{1/2}f \neq 0$ and g is in the domain of H_0, then

$$\text{meas} \, sp(U) \geqq 2\pi[1 + 2 \, ||f||^2 (H_0 g, g)/|| g ||^4]^{-1}. \qquad (13)$$

The proof of the Lemma was given in [14].

2. Theorem. *Let H_0 and H_1 be defined on $L^2(-\infty, \infty)$ by (11), and suppose that $V(x)$ is continuous on $-\infty < x < \infty$, satisfies*

$$0 \leqq V(x) \leqq \text{const.} \quad \text{on} \quad -\infty < x < \infty, \tag{14}$$

and that

$$\int_{-\infty}^{\infty} V(x)\, dx < \infty. \tag{15}$$

Then H_0 and H_1 are absolutely continuous and the strong limits U_+ and U_- of (9) exist as unitary operators which satisfy (10). If, in addition,

$$\lim_{b-a\to\infty} \inf (b-a)^{-3} \int_a^b V^{-1}(x)\, dx = 0, \tag{16}$$

then the spectrum of each operator U_+ and U_- is the entire unit circle $|z| = 1$.

Condition (16) rules out, for example, $V(x) = 0$, a function for which the assertion of the Theorem concerning the spectra of the associated operators U_+ and U_- is surely false. In fact, by (9), $U_+ = I$ and $U_- = I$, hence also, corresponding to (3), $U_+^\epsilon = I$ and $U_-^\epsilon = I$ for all ϵ.

It is easily verified that an example of a function satisfying all the conditions (14)-(16) is furnished by $V(x) = |x|^{-c}$, $1 < c < 2$ ($c = $ const.), when x is large, and where $V(x)$ is defined so as to be continuous and non-negative near $x = 0$.

3. Proof of the theorem. It is known that H_0 is absolutely continuous with the spectrum $0 \leqq \lambda < \infty$ (see Weyl [18], Kodaira [11], Titchmarsh [16], p. 59). Relation (14) guarantees that the spectrum of H_1 is contained in the half-line $0 \leqq \lambda < \infty$, while (15) implies via, for instance, asymptotic formulas for the solutions of the equation $y'' + (\lambda - V(x))y = 0$, that H_1 is absolutely continuous with the spectrum $0 \leqq \lambda < \infty$; see Wintner [17], p. 421. Relations (14) and (15) imply also that $V(x)$ is of class $L^2(-\infty, \infty)$. An application of a result of Kuroda [12] (see Theorem 3.1, pp. 438-439 and Theorem 5.1, p. 21; also a reference cited on p. 21 to T. Ikebe) then implies the existence of the strong limits (9) as unitary operators satisfying (10).

In order to complete the proof of the Theorem it will be shown that (16) implies

$$\inf_g \left[\int_{-\infty}^{\infty} g'^2\, dx \int_{-\infty}^{\infty} V^{-1}(x)g^2\, dx \Big/ \left(\int_{-\infty}^{\infty} g^2\, dx \right)^2 \right] = 0, \tag{17}$$

where g, g'' (hence g') and $V^{-1/2}g$ belong to $L^2(-\infty, \infty)$. Since

$$(H_0 g, g) = -\int_{-\infty}^{\infty} gg''\, dx = \int_{-\infty}^{\infty} g'^2\, dx,$$

it is seen that (17) implies meas $sp(U_\pm) = 2\pi$ by virtue of (13). There remains then to verify (17).

To this end, consider the function $y = z(x)$ defined on $-\infty < x < \infty$ by $z(x) = (b-a)^{-3/2}(x-a)$ for $a \leqq x \leqq (a+b)/2$, $z(x) = z(a+b-x)$ for $(a+b)/2 \leqq x \leqq b$, and $z(x) = 0$ for x outside the interval $a = x = b$. It is seen that $z(x)$ is continuous, has a piecewise continuous first derivative, and that

$$\int_{-\infty}^{\infty} z^2\, dx = 1/12, \qquad \int_{-\infty}^{\infty} z'^2\, dx = (b-a)^{-2},$$

and

$$\int_{-\infty}^{\infty} V^{-1} z^2 \, dx = \int_{a}^{b} V^{-1} z^2 \, dx \leqq \int_{a}^{b} V^{-1} \, dx / 4(b - a).$$

Hence, the expression $[\cdots]$ of (17) is, for $g = z$, majorized by const. $(b - a)^{-3} \int_{a}^{b} V^{-1} \, dx$. It is clear that each z can be smoothed out so as to obtain a function g possessing continuous second derivatives of the type allowed in (17) and such that $[\cdots]$ has again the same majorant. Condition (16) now yields (17) and the proof of the Theorem is complete.

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AN UPPER BOUND ON NON-NEGATIVE TRANSIENT RESPONSES*

BY

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In a recent note [1], it was shown that, if the real-valued function $w(t)$ of the real variable t is zero for $t < 0$ and if its Laplace transform $W(s)$ is given by

$$W(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{s^m + b_{m-1} s^{m-1} + \cdots + b_0} = \frac{N(s)}{D(s)}, \tag{1}$$

where $m \geq 2n$ and the real parts of the roots of the polynomial $D(s)$ are all non-positive,

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