CONDUCTION-CONVECTION FROM A CYLINDRICAL SOURCE WITH INCREASING RADIUS*

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Summary. The problem of heat flow by conduction and convection from a cylindrical source with increasing radius is solved. A quasi stationary state solution is obtained for the case of a finite convection coefficient and with the radius increasing at a constant velocity. A transient solution is obtained for the case of an infinite convection coefficient and with the radius increasing at a rate proportional to the square root of time.

In the latter case an explicit evaluation of an integral form of the solution is obtained by showing that the solution, in a certain coordinate system, is independent of time and thus the partial differential equation reduces to an ordinary differential equation which is solved explicitly.

Introduction. The theory of heat flow from a moving source is of interest in a number of applications; some of these are discussed by Crank [5]†. Most of these applications are concerned with heat conduction from a moving plane source.

The problem of heat flow from a cylindrical source with increasing radius has been studied by H. R. Bailey, B. K. Larkin and H. Ramey, [1], [2] and [9]. This problem arises in connection with a secondary oil recovery process by underground combustion. The

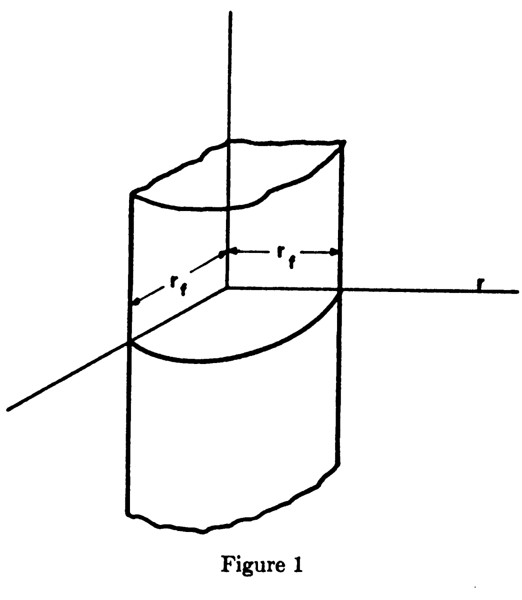

Figure 1

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†Numbers in brackets refer to the references listed at the end of the paper.

papers mentioned above are mostly concerned with the conduction mechanism for heat flow; however, Ramey [9] does give an approximation of the convection effects. Heat flow by conduction and convection, without moving sources, has been studied by Bland [4] in connection with heat exchanger applications.

In the present paper we consider heat flow by conduction and convection from a cylindrical heat source with increasing radius, $r = r_f$, as shown in Fig. 1; gas (air) is injected at $r = 0$ into a porous solid of infinite extent and it is the oxygen from this air that supports the combustion. With certain simplifying assumptions (see Bailey and Larkin [3]) the equations describing this process can be written in the form

$$\frac{\partial^2 T_s}{\partial r^2} + \frac{1}{r}\frac{\partial T_s}{\partial r} - a^2 \frac{\partial T_s}{\partial t} + k^{-1}\Phi(r, t) = k^{-1}h(T_s - T) = \frac{P(t)}{r}\frac{\partial T}{\partial r}, \tag{1}$$

where T_s is the temperature of the solid, T is the temperature of the gas, k is the conductivity of the solid, a^2 is the reciprocal diffusivity, h is the convection coefficient, r is the radius, t is time, $\Phi(r, t)$ is a source function and $P(t)$ is a function of time whose form depends on the air injection program. From the second equality in system (1), we have

$$T_s = T + \frac{kP(t)}{hr}\frac{\partial T}{\partial r} \equiv T + E(r, t)\frac{\partial T}{\partial r},$$

where the identity is a definition of $E(r, t)$. With T_s replaced by the above expression, system (1) can be written as a single partial differential equation:

$$\frac{\partial^2(T + E\,\partial T/\partial r)}{\partial r^2} + \frac{1}{r}\frac{\partial(T + E\,\partial T/\partial r)}{\partial r} - a^2\frac{\partial(T + E\,\partial T/\partial r)}{\partial t} + k^{-1}\Phi(r, t) - k^{-1}hE\frac{\partial T}{\partial r} = 0. \tag{2}$$

The functions $P(t)$ and $\Phi(r, t)$ depend on the assumed air injection program; two cases are of particular interest in underground combustion: (a) constant velocity, $r_f = v_f t$, where v_f is a constant, in this case it is shown in [3] that $P(t) = k_1 v_f^2 t$ and $\Phi(r, t) = qv_f\delta(r - r_f)$, (b) constant injection rate, $r_f^2 = 2Ut$, where U is a constant, in this case it is shown in [3] that $P(t) = 2n$ where n is a constant and $\Phi(r, t) = qUr_f^{-1}\delta(r - r_f)$. In the above equation k_1 and q are constants and $\delta(r - r_f)$ is the Dirac delta function.

In Sec. 2 of this paper a quasi-stationary state solution of Eq. (2) is obtained for the constant velocity case. Solutions of Eq. (1) are obtained in Sec. 3 for an instantaneous cylindrical source and for a cylindrical source with increasing radius (i.e., the constant injection rate case). This latter solution is obtained in the form of an integral which is evaluated explicitly in Sec. 4.

2. Quasi-stationary state solution for the constant velocity case. In this section a quasi-stationary state solution of Eq. (2) is obtained for the constant velocity case, i.e., $r_f = v_f t$ with v_f a positive constant. Clearly there would not be a steady state solution in the usual sense; however, it is assumed that an observer moving with the source would see steady state attained for sufficiently large values of time. This assumption is discussed by Jacob [7].

Putting $r = s + v_f t$ transforms Eq. (2) into

$$\frac{\partial^2(T + E\,\partial T/\partial s)}{\partial s^2} + \frac{1}{s + v_f t}\frac{\partial(T + E\,\partial T/\partial s)}{\partial s} + a^2 v_f \frac{\partial(T + E\,\partial T/\partial s)}{\partial s} - a^2 \frac{\partial(T + E\,\partial T/\partial s)}{\partial t} - \frac{hE}{k}\frac{\partial T}{\partial s} + \frac{q v_f}{k}\delta(s) = 0, \tag{3}$$

where $E = k\,P(t)/h(s + v_f t) = kk_1\,v_f^2 t/h(s + v_f t)$. The variable s measures the radial distance from the moving source, and the assumption that quasi-stationary state is attained means that $T(s, t)$ does not change with respect to time for sufficiently large and finite s. Thus, passing to the limit as $t \to \infty$, Eq. (3) is reduced to the ordinary differential equation,

$$E\frac{d^3T}{ds^3} + \frac{d^2T}{ds^2} + a^2 E v_f \frac{d^2T}{ds^2} + a^2 v_f \frac{dT}{ds} - k_1 v_f \frac{dT}{ds} = 0, \tag{4}$$

where the last term in Eq. (3) is replaced by the equivalent boundary condition, (a), below. The term $k_1 v_f\, dT/ds$ is the limiting form of the corresponding term $hE/k\; \partial T/\partial s$ in Eq. (3), this follows from the definitions in Sec. 1 of $E(r, t)$ and $P(t)$ for the constant velocity case.

The boundary conditions are:

(a) $$\left.\frac{dT_s}{ds}\right|_{s=0+} - \left.\frac{dT_s}{ds}\right|_{s=0-} \equiv \left.\frac{dT}{ds}\right|_{s=0+} - \left.\frac{dT}{ds}\right|_{s=0-} + E\left.\frac{d^2T}{ds^2}\right|_{s=0+} - E\left.\frac{d^2T}{ds^2}\right|_{s=0-} = \frac{-qv_f}{k},$$

(b) $$\left.\frac{dT}{ds}\right|_{s=0+} = \left.\frac{dT}{ds}\right|_{s=0-},$$

(c) T remains bounded as $s \to -\infty$,

(d) $T \to 0$ as $s \to -\infty$,

where $dT/ds\,|_{s=0+}$ and $dT/ds\,|_{s=0-}$ are the right and left hand derivatives respectively at $s = 0$.

Condition (a) is equivalent to the source term in Eq. (3). Condition (b) results from the assumption that $\partial T/\partial s$ is continuous at $s = 0$, i.e., that the heat source is only in the solid phase. Conditions (c) and (d) follow immediately from heat balance considerations for a moving source.

The general solution of Eq. (4) is of the form $T = C_0 + C_1 \exp(r_1 s) + C_2 \exp(r_2 s)$ and it can be seen from the characteristic equation determining r_1 and r_2 that they are both real and negative, provided k_1 is less than a^2 which is the case in underground combustion. Thus, in order to satisfy boundary condition (c) for $s < 0$, we must have $C_1 = C_2 = 0$. For $s > 0$, and to satisfy boundary condition (d), we must have $C_0 = 0$. Imposing boundary conditions (a) and (b) results in the following solution of (4) subject to conditions (a), (b), (c) and (d).

$$T = \frac{Rq}{(R - S)a^2 k}, \qquad s \le 0$$

$$T = \frac{2Rq}{a^2 kW}\left\{\frac{\exp[(-1 - R + W)s/2E]}{1 + R - W} - \frac{\exp[(-1 - R - W)s/2E]}{1 + R + W}\right\}, \quad s \ge 0 \tag{5}$$

where $R = a^2 E v_f$, $S = k_1 E v_f$, $W = [(1 - R)^2 + 4S]^{1/2}$.

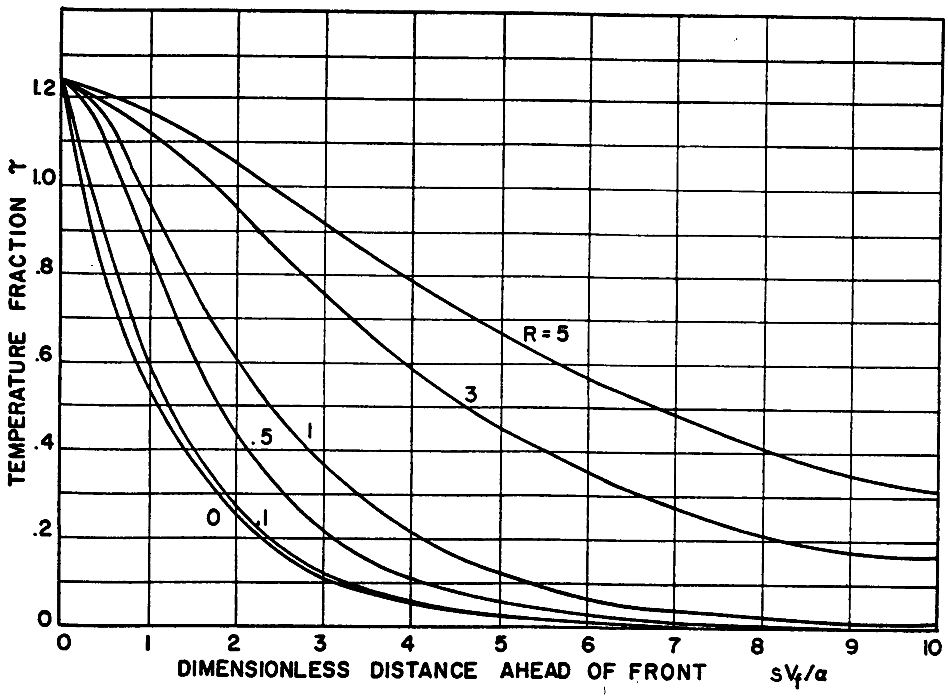

Figure 2.

The above solution is also a solution for the corresponding linear problem of a plane source moving at a constant velocity in a semi-infinite medium. This follows from the observation that a cylindrical source approaches a plane source for sufficiently large radii.

If it is assumed that conduction is the only mechanism for heat transfer then the corresponding formula for the temperature has been obtained [1] by determining the limit as $t \to \infty$ of an explicit solution of the partial differential equation. The latter method is more difficult; however, it is more satisfying since it is not necessary to make the *a priori* assumption that a quasi-stationary state will be obtained.

It is shown in [3] that $R = 5.18\ S$ if the injection gas is air; Fig. 2 shows the temperature profiles for this case. The temperatures are given in terms of the temperature fraction, $\tau = T/(q/a^2k)$.

3. Transient solutions for the constant injection rate case. In this section explicit solutions of Eq. (2) are obtained for the constant injection rate case, $r_f^2 = 2Ut$, assuming infinite convection coefficient, h. With these additional assumptions we have $E = 0$, and $P(t) = 2n$, where $n > 0$ is a constant depending on the parameters in the physical problem. Equation (2) thus takes the form

$$\frac{\partial^2 T}{\partial r^2} + \frac{1 - 2n}{r} \frac{\partial T}{\partial r} - a^2 \frac{\partial T}{\partial t} + k^{-1}\Phi(r, t) = 0. \tag{6}$$

In the following analysis two forms of the source function, Φ, are considered: (1) an instantaneous cylindrical source, (2) a continuous cylindrical source with increasing radius.

3.1. *Instantaneous cylindrical source.* Consider a cylindrical source with radius r_0

which liberates one unit of heat per unit area at time t_0. In this case Φ can be represented in terms of Dirac delta functions and Eq. (6) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1 - 2n}{r} \frac{\partial T}{\partial r} - a^2 \frac{\partial T}{\partial t} + k^{-1} \delta(r - r_0) \delta(t - t_0) = 0. \tag{7}$$

The substitution $T = Sr^n$ transforms the above equation into

$$\frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} - \frac{n^2}{r^2} S - a^2 \frac{\partial S}{\partial t} + k^{-1} r^{1-n} \delta(r - r_0) \delta(t - t_0) = 0.$$

Let R be the Hankel transform of S, e.g., [10], i.e.,

$$R(\xi) = \int_0^\infty r J_n(\xi r) S(r, t) \, dr.$$

Then taking the Hankel transform of the above equation gives

$$\begin{aligned} \xi^2 R - a^2 \frac{dR}{dt} &= -k^{-1} \int_0^\infty r^{1-n} J_n(\xi r) \delta(r - r_0) \delta(t - t_0) \, dr \\ &= -k^{-1} r_0^{1-n} J_n(\xi r_0) \delta(t - t_0). \end{aligned}$$

Solving the above equation we obtain the particular solution

$$R = k^{-1} a^{-2} r_0^{1-n} J_n(\xi r_0) \exp [\xi^2 (t - t_0)/a^2]$$

and taking the inverse Hankel transform, e.g., [6], gives

$$\begin{aligned} S(r, t) &= \int_0^\infty \xi R(\xi, t) J_n(\xi r) \, d\xi \\ &= 2^{-1} k^{-1} (t - t_0)^{-1} r_0^{1-n} \exp [-a^2 (r^2 + r_0^2)/4(t - t_0)] I_n[r r_0 a^2/2(t - t_0)]. \end{aligned}$$

Finally, from the transformation relating S and T, the following formula for the temperature is obtained:

$$T_{ic} = 2^{-1} k^{-1} (t - t_0)^{-1} r_0^{1-n} r^n \exp [-a^2 (r^2 + r_0^2)/4(t - t_0)] I_n[r r_0 a^2/2(t - t_0)], \tag{8}$$

where the subscripts ic have been added to T to indicate temperature due to an instantaneous cylindrical source.

For $n = 0$ the above expression for T_{ic} reduces to the known, [1], solution for conduction only. For $n > 0$, it is indicated below that the solution has the required properties.

It can be shown by direct computation that T_{ic}, for $t > 0$, satisfies

$$\frac{\partial^2 T}{\partial r^2} + \frac{1 - 2n}{r} \frac{\partial T}{\partial r} - a^2 \frac{\partial T}{\partial t} = 0.$$

Also, from the asymptotic form of $I_n(z)$ for large z, it can be seen that $T_{ic} \to 0$ when $t \to t_0$ at all points except on the cylinder $r = r_0$ where T_{ic} becomes infinite. By using a generalization of Webber's first integral [11],

$$\int_0^\infty \exp (-a^2 u^2) J_n(bu) u^{n+1} \, du = \frac{b^n}{(2a^2)^{n+1}} \exp (-b^2/4a^2),$$

we have for $t > t_0$

$$a^2 k \int_0^\infty 2\pi r T_{ic}(r, t; r_0, t_0) \, dr = 2\pi r_0$$

which is the amount of heat liberated per unit length by the instantaneous source.

3.2. *Cylindrical source with increasing radius.* It is reasonable, [2, 3], to assume in an underground combustion process that the source function can be expressed by the formula

$$\Phi(r, t) = q\,\delta(r - r_f)\,dr_f/dt,$$

where $r_f(t)$ is the radius of the cylindrical source, δ is the Dirac delta function and q is constant. The case of most practical interest is with constant air injection rate; and in this case the position of the front is given by the formula, $r_f^2 = 2Ut$, where U is a constant.

With the source function described above Eq. (6) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1 - 2n}{r}\frac{\partial T}{\partial r} - a^2\frac{\partial T}{\partial t} + \frac{qU}{kr_f}\,\delta(r - r_f) = 0. \tag{9}$$

The solution of the above equation with $T = 0$ at $t = 0$ is given in terms of T_{ic}, Eq. (8), by the formula

$$T(r, t) = qU\int_0^t\int_0^\infty T_{ic}(r, t; r_0, t_0)r_f^{-1}\,\delta(r - r_f)\,dr_0\,dt_0,$$

where r_f is evaluated at t_0, i.e., $r_f = (2Ut_0)^{1/2}$. Performing the indicated integration with respect to r_0 we obtain

$$T = 2^{-1}k^{-1}qUr^n\int_0^t(2Ut_0)^{-n/2}(t - t_0)^{-1} \cdot\exp\left[-a^2(r^2 + 2Ut_0)/4(t - t_0)\right]I_n[2^{-1}(t - t_0)^{-1}a^2r(2Ut_0)^{1/2}]\,dt_0. \tag{10}$$

4. An explicit evaluation of the integral in Eq. (10). An explicit evaluation of the above integral, Eq. (10), is obtained by applying a method introduced in [1] for the case of conduction only. Making the substitutions $y = r/r_f$ and $\tau = (t - t_0)/t$, Eq. (10) is transformed into

$$T(y, t) = 2^{-1}k^{-1}qUy^n\int_0^1\tau^{-1}(1 - \tau)^{-n/2}\exp\left[-a^2U(y^2 + 1 - \tau)/2\tau\right] \cdot I_n[\tau^{-1}a^2Uy(1 - \tau)^{1/2}]\,d\tau. \tag{11}$$

Thus $T(y, t)$ is independent of t since t does not appear in the right side of the above equation.

The transformation $y = r/r_f$ in the partial differential equation (6) gives

$$\frac{\partial^2 T}{\partial y^2} + \frac{1 - 2n}{y}\frac{\partial T}{\partial y} + a^2Uy\frac{\partial T}{\partial y} - 2a^2Ut\frac{\partial T}{\partial t} = 0,$$

where the source term is replaced by boundary condition (b) below. Since we have shown that $T(y, t)$ is independent of t we have $\partial T(y, t)/\partial t = 0$ and the above equation becomes

$$\frac{d^2T}{dy^2} + \frac{1 - 2n}{y}\frac{dT}{dy} + 2By\frac{dT}{dy} = 0 \tag{12}$$

where $B = a^2U/2$.

The boundary conditions are

$$\text{(a)}\quad T = T_0 \quad \text{at} \quad y = 0$$

$$\text{(b)}\quad \left.\frac{dT}{dy}\right|_{y=1+} - \left.\frac{dT}{dy}\right|_{y=1-} = -qU/k$$

$$\text{(c)}\quad T \to 0 \quad \text{as} \quad y \to \infty .$$

Condition (a) corresponds to the inlet temperature of the injected gas. It should be noted for the solution given by Eq. (11) that $T_0 = 0$, and thus we have shown that $T(y, t)$ is independent of t only for $T_0 = 0$. We shall assume that $T(y, t)$ is independent of t for any value of T_0 and check the resulting answer to see that it satisfies the partial differential equation and the boundary conditions. Condition (b) replaces the source term, $k^{-1}\,\Phi(r, t) = qUk^{-1}r_f^{-1}\,\delta(r - r_f)$; that is the heat source at $r = r_f$ is equivalent to a discontinuity in the temperature derivative at $r = r_f$.

The ordinary differential Equation (12) can be solved either by making the substitutions $z = By^2$ and $T = We^{-z}$ which reduce Eq. (12) to a confluent form of the hypergeometric equation or by the substitution $w = \partial T/\partial z$ and integrating the resulting first order equation. The general solution of (12) is given by

$$\begin{aligned} T &= C_2 + C_3\Gamma(n, By^2), \qquad y \geq 1 \\ T &= K_2 + K_3\Gamma(n, By^2), \qquad y < 1 \end{aligned} \tag{13}$$

where $\Gamma(a, b)$ is the incomplete gamma function defined by the integral

$$\Gamma(a, b) = \int_b^\infty e^{-u}u^{a-1}\,du.$$

Values of the incomplete gamma function have been tabulated, for example see Pearson [8]. The constants C_2 , C_3 , K_2 and K_3 are determined by the boundary conditions (a), (b) and (c), and the requirement that T be continuous at $y = 1$. These conditions lead to the following relations respectively:

(a) $T_0 = K_2 + K_3\,\Gamma(n)$ where $\Gamma(n)$ is the gamma function
(b) $-\,2C_3e^{-B}B^n + 2K_3e^{-B}B^n = -\,qU/K$
(c) $C_2 = 0$

(continuity at $y = 1$) $C_2 + C_3\,\Gamma(n, B) = K_2 + K_3\,\Gamma(n, B)$.

Solving the above equations for C_2 , C_3 , K_2 and K_3 and combining with Eq. (13) we obtain

$$\begin{aligned} T &= \frac{e^BB^{-n}\Gamma(n, By^2)}{\Gamma(n)}\,[T_0e^{-B}B^n + 2^{-1}k^{-1}qU(\Gamma(n) - \Gamma(n, B))], \qquad y \geq 1 \\ T &= e^BB^{-n}qU2^{-1}k^{-1}\Gamma(n, B) + \frac{\Gamma(n, By^2)}{\Gamma(n)}\,[T_0 - qU2^{-1}k^{-1}e^BB^{-n}\Gamma(n, B)], \quad y \leq 1. \end{aligned} \tag{14}$$

It can be shown by direct computation that the above expressions for T (with y replaced by r/r_f , $r_f^2 = 2Ut$) satisfy the partial differential equation (9) and the conditions $T = 0$ at $t = 0$, $T = T_0$ at $r = 0$ and $T \to 0$ as $r \to \infty$.

Figure 3 has been obtained by evaluating Eq. (14) for the case $T_0 = 0$ and $r/r_f = 1$. The fractional temperature rise, $\tau = T/(q/a^2K)$, is plotted as a function of a dimension-

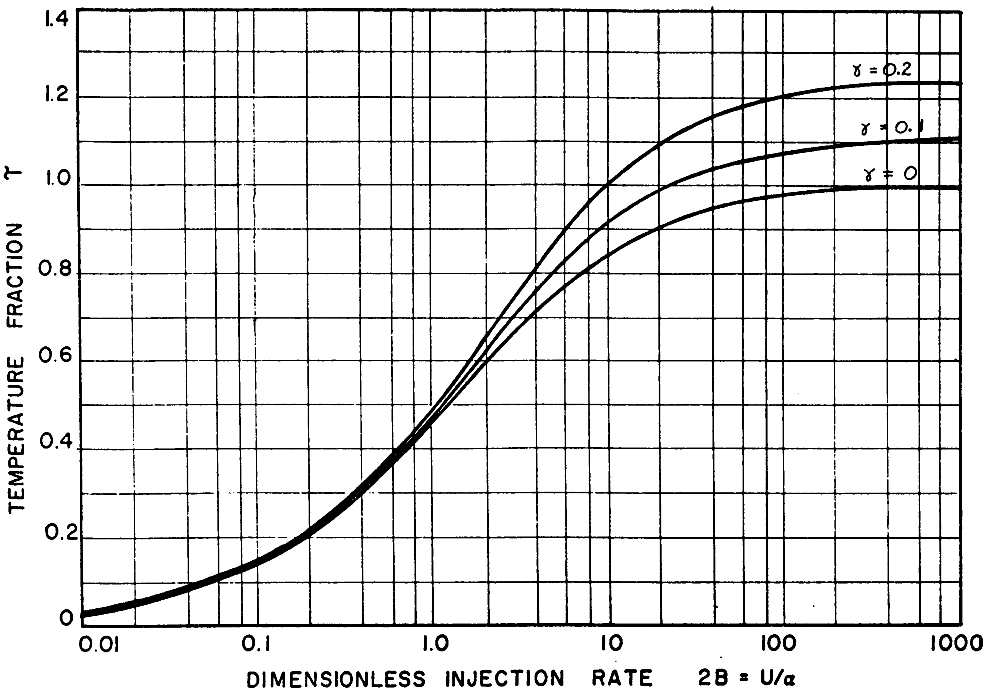

Figure 3.

less injection rate, $2B = a^2U$, for various values of a relative gas velocity $\gamma = n/B$. The curve $\gamma = 0$ corresponds to assuming conduction only. The value $\gamma = 0.2$ is typical for an underground combustion process.

Remarks. (1) For $n = 0$ the Equation (14) reduces to corresponding result, [1], for the conduction only case. In this case the terms in T_0 do not appear, and thus the temperature at $r = 0$ cannot be prescribed but is determined by the solution. For $n > 0$ the temperature at $r = 0$ is prescribed and corresponds to the inlet temperature of the injected gas.

(2) Since the solution is unique, we may equate the two solutions given by Eqs. (11) and (14) (for $T_0 = 0$) and obtain the following evaluation

$$\int_0^1 \frac{d\tau}{\tau(1-\tau)^{n/2}} \exp\left[-B(y^2+1)/\tau\right] I_n\left(\frac{2yB(1-\tau)^{1/2}}{\tau}\right)$$

$$= \begin{cases} B^{-n}y^{-n}\Gamma(n, By^2)\left[1 - \dfrac{\Gamma(n,B)}{\Gamma(n)}\right], & y \geq 1 \\ B^{-n}y^{-n}\Gamma(n, B)\left[1 - \dfrac{\Gamma(n,By^2)}{\Gamma(n)}\right], & y \leq 1. \end{cases}$$

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