ON THE FLOW OF A VISCOUS ELECTRICALLY CONDUCTING FLUID*

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The equations governing the steady two-dimensional flow of a conducting fluid whose free stream direction is parallel to an applied magnetic field can be linearized, by an extension of the Oseen treatment of viscous flows, and reduced to the dimensionless form

$$\Delta\left(\Delta\psi - \frac{\partial\psi}{\partial x} + \beta\,\frac{\partial A}{\partial x}\right) = 0, \tag{1}$$

$$\Delta A - \epsilon\left(\frac{\partial A}{\partial x} - \frac{\partial\psi}{\partial x}\right) = 0. \tag{2}$$

Here the stream function, $\psi(x, y)$ and vector magnetic potential $A(x, y)$ are defined by the relations

$$\mathbf{q} = \nabla \times \psi(x, y)\mathbf{i}_3$$

$$\mathbf{H} = \nabla \times A(x, y)\mathbf{i}_3\ .$$

In addition, $\beta = (\mu H_0^2)/\rho V_0^2)$, and $\epsilon = \sigma\mu v$. (H_0 and V_0 are the free stream values of magnetic intensity and fluid velocity; the other symbols have their usual meanings.). The details of the reduction can be found in Ref. [1].

A transform analysis of the flow past a rigid impermeable semi-infinite flat plate, [1], results in a solution of the foregoing equations which is of the form

$$\begin{aligned} \psi &= \xi f(\eta), \\ A &= \xi g(\eta), \end{aligned} \tag{3}$$

with

$$(\xi + i\eta)^2 = x + iy.$$

Equations (1) and (2) admit general solutions of the form expressed in (3) and an entire class of parabolic flow problems is then explicitly soluble in terms of known elementary functions. Typical among these would be the flow in an aligned magnetic field past a parabolic cylinder which may be injecting another conducting fluid into the main stream. At least three different regions and possibly five independent parameters are involved in this complex interaction and an exact numerical solution of the non-linear equations would require an immense amount of work. The linear theory, however, affords an easy way of examining the qualitative effects of a variation of any particular parameter on the fluid motion or magnetic field because explicit analytical solutions are readily obtainable. In other words, a rapid qualitative survey of any particular aspect of the phenomenon can easily be made using the linear theory and this can then be followed by a quantitative analysis if the results warrant a more exact investigation.

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The substitution of the functional forms of Eqs. (3) into (1) and (2) yield

$$\frac{-4\xi\eta}{(\xi^2+\eta^2)^2}\frac{d}{d\eta}\left[\frac{f''}{2}-f+\eta f'+\beta(g-\eta g')\right] + \frac{\xi}{\xi^2+\eta^2}\frac{d^2}{d\eta^2}\left[\frac{f''}{2}-f+\eta f\ +\beta(g-\eta g')\right]=0 \tag{4}$$

and

$$\tfrac{1}{2}g''+\epsilon(f-\eta f'-g+\eta g')=0. \tag{5}$$

The former equation is satisfied (fortuitously) if

$$\frac{d}{d\eta}\left[\tfrac{1}{2}f''-f+\eta f'+\beta(g-\eta g')\right]=0$$

i.e.,

$$\tfrac{1}{2}f''-f+\eta f'+\beta(g-\eta g')=K, \tag{6}$$

where K is a constant. The general solution of Eqs. (5) and (6) is, for $\beta \le 1$ (super-Alfvén flow)

$$f(\eta)+\alpha_1 g(\eta)=a_0\eta+a_1\frac{G(\Omega\eta)}{\Omega}-\frac{K}{\Omega^2}, \tag{7}$$

$$f(\eta)+\alpha_2 g(\eta)=b_0\eta+b_1\frac{G(\omega\eta)}{\omega}-\frac{K}{\omega^2}, \tag{8}$$

where

$$G(\eta)=\eta\operatorname{erf}\eta+\frac{\exp(-\eta^2)}{\pi^{1/2}},$$

$$2\Omega^2=1+\epsilon-\lambda,\quad 2\omega^2=1+\epsilon+\lambda,\quad \lambda=((1-\epsilon)^2+4\epsilon\beta)^{1/2},\quad 2\epsilon\alpha_{1,2}=1-\epsilon\pm\lambda,$$

and a_0, a_1, b_0, b_1, K are all arbitrary constants. If there is more than one parabolic domain i.e. body plus fluid, the general solutions appropriate in each region must be joined by matching conditions at the boundaries.

As a particular example consider the flow past a semi-infinite flat plate along which the horizontal velocity component is zero and the vertical component is $-\partial\psi(x,0)/\partial x = -V/2x^{\frac{1}{2}}$, $x \ge 0$ (the constant V is negative for injection and positive for suction). The boundary conditions require that $f(0)=V$, $g(0)=f'(0)=0$ and $f'(\infty)=2$. The solution is then

$$f(\eta)=2\eta+\left[2\omega\alpha_2+\pi^{1/2}\frac{V(\Omega^2-\omega^2)\alpha_1\alpha_2}{\alpha_1-\alpha_2}\right](\alpha_1\Omega-\alpha_2\omega)^{-1}\frac{H(\Omega\eta)}{\Omega} - \left[2\Omega\alpha_1+\pi^{1/2}\frac{V(\Omega^2-\omega^2)}{\alpha_1-\alpha_2}\alpha_1\alpha_2\right](\alpha_1\Omega-\alpha_2\omega)^{-1}\frac{H(\omega\eta)}{\omega}, \tag{9}$$

$$g(\eta)=2\eta-(2\omega+\pi^{1/2}V(\Omega^2-\omega^2)\alpha_1)(\alpha_1\Omega-\alpha_2\omega)^{-1}\frac{H(\Omega\eta)}{\Omega} + (2\Omega+\pi^{1/2}V(\Omega^2-\omega^2)\alpha_2)(\alpha_1\Omega-\alpha_2\omega)^{-1}\frac{H(\omega\eta)}{\omega}, \tag{10}$$

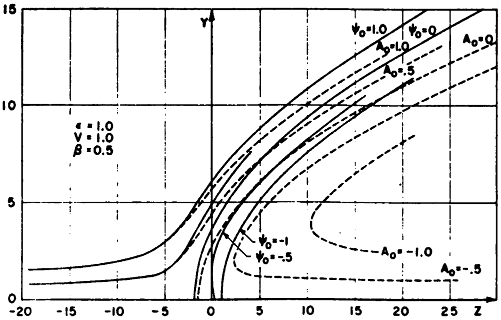

FIG. 1. The flow past a flat plate which is injecting fluid into the mainstream. Solid curves represent streamlines and dashed curves are field lines. The injected fluid is the same as that of the free stream.

where

$$H(\eta) = \eta \operatorname{erfc} \eta + \pi^{-1/2}(1 - \exp - \eta^2).$$

A typical injection flow pattern is shown in Fig. 1. The two distinct flow regimes which occur are separated by the zero streamline $\psi = \xi f(\eta) = 0$. (That is, $\xi = 0$ and $x + \bar{\eta}^2 = \frac{1}{4}y^2\bar{\eta}^{-2}$ where $f(\bar{\eta}) = 0$.) The interior domain is occupied by the injected fluid and the exterior realm consists of the fluid from the free stream. Two distinct magnetic fields also occur; an induced field occupies the domain interior to $\xi g(\eta) = 0$ and the applied field occupies the exterior region. Descriptively speaking, the free stream lines are pushed outward by the emerging fluid and drag the magnetic field lines behind them. The induced interior magnetic field closely resembles the field of a semi-infinite solenoid of current.

The solution of the more general problem involving the flow past a parabolic cylinder (which may be injecting or drawing fluid or impermeable) can be solved in as straightforward a manner as the preceding example. The general solutions, valid in each region, must be properly joined at the interface boundaries. Across the parabolic interface separating two fluids it is required that the fluid velocity, viscous stress, tangential component of magnetic intensity, $\nabla \times A\mathbf{i}_3$, and normal component of magnetic field $\mu\nabla \times A\mathbf{i}_3$ be continuous. (The latter two conditions hold across any surface.) This corresponds to requiring the continuity of f, f', f'', μg, and g'. The tangential velocity component at the surface of a body is zero although the normal component may be prescribed to simulate injection or suction. (It is necessary, however, to maintain a parabolic flow.) The effects of magnetization can also be examined by prescribing a non-zero boundary condition on $g(0)$. The arbitrary constants are easily evaluated and the solution is explicitly determined in terms of elementary functions, the most complicated of which is erf η. The location of an interface such as that between injected fluid and free stream is obtained from the condition that it constitutes part of the zero stream line $\xi f(\eta) = 0$.

A particular solution, worth quoting because of its simplicity, is that for the perfectly conducting flow past the impermeable parabolic cylinder $\eta = \eta^*$

$$f(\eta) = g(\eta) = 2(\eta - \eta^*) - \frac{2(1-\beta)^{-1/2}}{\operatorname{erfc}(1-\beta)^{1/2}\eta^*}\,[H(1-\beta)^{1/2}\eta) - H((1-\beta)^{1/2}\eta^*)].$$

The first term of the right hand side represents the inviscid solution and the second is the viscous magnetohydrodynamic correction.

The linear theory is, of course, grossly inaccurate near stagnation points of either fluid or magnetic field. Other sources of extreme errors may arise because the actual magnetic force is replaced by its component normal to the free stream direction. This, for infinite conductivities, requires shear discontinuities to maintain the dynamic equilibrium of current carrying interfaces. It should also be noted that the linearized version of the infinite conductivity condition $\mathbf{q} \times \mathbf{H} = 0$ need not require the alignment of field and fluid if they are not already so directed somewhere within the perfectly conducting region. If the linear theory is not applied to problems for which it is obviously inappropriate, fairly good qualitative results can be expected.

Substantially correct quantative results may also be obtainable by rescaling the formulae developed from this theory in accordance with the modification introduced by Lewis and Carrier [2].

References

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2. F. A. Lewis and G. F. Carrier, *Some remarks on the flat plate boundary layer*, Quart. Appl. Math. **7**, 228 (1949)