sphere. The region in the neighbourhood of $r = 1.70$ is consequently the region of maximum unsteadiness.

Acknowledgment. I am grateful to Dr. Ram Ballabh for supervising this work. I am also grateful to the referee for pointing out an error in the earlier version of the paper.

References

1. H. Lamb, *Hydrodynamics*, 6th ed., Cambridge Univ. Press, 1932, p. 588
2. A. Erdélyi and associates, *Tables of integral transforms*, Bateman Manuscript Project, vol. 1, McGraw-Hill, New York, 1954, p. 245 and p. 247.

DYNAMIC PROGRAMMING APPROACH TO OPTIMAL INVENTORY PROCESSES WITH DELAY IN DELIVERY*

BY RICHARD BELLMAN (*The RAND Corporation, Santa Monica, California*)

Summary. The usual dynamic programming approach to inventory processes with delays in delivery leads to functions of many variables. This multi-dimensionality prevents the straightforward utilization of digital computers.

Using a type of transformation previously applied in the study of engineering control processes, we show that a class of inventory processes with time lags can be treated in terms of sequences of functions of one variable, regardless of the length of the delay.

1. Introduction. The problem of determining ordering policies which minimize the cost of operating supply depots and stockrooms is one which has attracted a great deal of attention in industrial and military circles in recent years. An analytic approach to these questions by way of functional equation techniques was inaugurated by Arrow, Harris, and Marschak, in a now classic paper, [1]. These investigations were extended by Dvoretzky, Kiefer, and Wolfowitz, [7], and Bellman, Glicksberg, and Gross, [6]; see also [2], and the books by Whitin, [8], and Arrow, Karlin and Scarf, [9].

Although this approach can be used to obtain analytic and computational solutions of a variety of processes in which there is no delay between an order for an additional supply of items and the delivery of these items, this method runs into dimensionality difficulties when time lags of more than a stage or two occur. If there is a delay of d stages in filling an order, the state of the system at any time is characterized not only by the present stock level, but also by the quantities on order which will arrive one, two, $\cdots$, d stages in the future.

It thus appears that functions of d variables necessarily arise when point of regeneration methods are employed to treat these processes.

In several papers devoted to the study of control processes arising in the engineering world [3], [4], we have shown that in some fortunate situations certain preliminary trans-

*Received February 3, 1960.

formations enable us to bypass functions of large dimension. Problems which seemingly require functions of ten or twenty variables can actually be reduced to questions involving functions of one or two variables. This vast reduction in dimension renders a computational solution feasible. An example of this phenomenon in the field of scheduling is contained in our paper on the "caterer" problem, [5].

Here we shall consider the application of this type of transformation to inventory processes. The stochastic nature of the process introduces certain features of complication. Although under appropriate assumptions of linearity exact solutions can be obtained, we shall concentrate here only on making the problem susceptible to computational solution.

2. Description of process. In this simplified model of an inventory process, we consider the question of stocking a single item which is subject to a stochastic demand at certain specified times which we denote by $0, 1, \cdots, N$. Since there is a penalty attached to inability to supply these demands, quantities of this item are purchased throughout the process at these same time points. This outage or penalty cost is a prescribed function of the excess of demand over supply. We suppose that there is a time lag of d units between an item being ordered and its delivery.

In many cases, there is an additional cost incurred in storing items. Since this introduces no additional analytic difficulties, but does complicate the arithmetic, we shall ignore this cost in our future discussion.

Assuming that we are given the purchase cost, the outage cost, and the distribution of demand at any stage, we wish to determine ordering policies which minimize the total expected cost.

Complicated as this problem is, it is one level simpler than many which arise in practice in which the distribution of demand is not completely known. These will be discussed at another time; see Dvoretzky, Kiefer and Wolfowitz [7].

3. Analytic formulation. A discussion of a straightforward dynamic programming formulation of processes of the type described in the foregoing paragraph may be found in [2]. As mentioned above, this direct approach leads to dimensionality difficulties. Here we postpone the introduction of dynamic programming techniques until an appropriate point.

Let us define the following quantities:

a. x_n = the stock level at the nth stage, prior to the delivery of the quantity ordered at the nth stage and the demand at the nth stage.

b. y_n = the quantity ordered at the nth stage,

$$n = 0, 1, \cdots, N - d; \tag{1}$$

c. z_n = the demand at the nth stage, $n = 0, 1, \cdots, N$;

d. w_n = the actual cost of the first n stages, $n = 0, 1, \cdots, N$.

All four of these quantities are stochastic with the exception of x_0 and y_0. The quantity y_n depends upon the observed stock level and the quantities ordered at preceding stages, $y_{n-1}, \cdots, y_{n-d}$. It is assumed here that y_n must be determined before z_n is observed. It will be clear that either case could be discussed.

The following relations are then immediate,

$$x_{n+1} = x_n - z_n + y_{n-d}, \tag{2}$$

where $y_k = 0$ for $k < 0$ and $k > N - d$, and

$$w_{n+1} = w_n + g(y_{n+1}) + \phi(x_{n+1}), \qquad n = 0, 1, \cdots . \tag{3}$$

Here $g(y)$ measures the cost of ordering y items and $\phi(x)$ is the cost of an outage of x items. A negative stock level is equivalent to outage.

It follows that

$$w_N = \sum_{i=0}^{N-d} g(y_i) + \sum_{k=0}^{N} \phi(x_k), \tag{4}$$

where the x_i are given by

$$\begin{aligned} x_i &= x_0 - (z_0 + z_1 + \cdots + z_{i-1}), \qquad i = 1, 2, \cdots, d-1, \\ x_i &= x_0 - (z_0 + z_1 + \cdots + z_{i-1}) + y_0 + y_1 + \cdots + y_{i-d}, \\ &\qquad\qquad i = d, \cdots, N. \end{aligned} \tag{5}$$

Substituting, we see that the total cost of an N-stage process may be written

$$\begin{aligned} w_N = \sum_{i=0}^{N-d} g(y_i) + \sum_{k=0}^{d-1} \phi(x_0 - (z_0 + z_1 + \cdots + z_{k-1})) \\ + \phi(x_0 - (z_0 + z_1 + \cdots + z_{d-1}) + y_0) \\ + \phi(x_0 - (z_0 + z_1 + \cdots + z_d) + y_0 + y_1) \\ + \cdots + \phi(x_0 - (z_0 + z_1 + \cdots + z_{N-1}) \\ + y_0 + y_1 + \cdots + y_{N-d}). \end{aligned} \tag{6}$$

We now wish to determine the policy functions, $\{y_i\}$, stochastic functions of the z_i, which will minimize the expected total cost. This expected value is taken over the random variables, $\{z_i\}$, independent random variables with a known distribution function.

4. Dynamic programming formulation. It is clear that in seeking the policy functions which minimize Exp (w_N), we can ignore the term

$$\operatorname*{Exp}_{z} \left[\sum_{k=0}^{d-1} \phi(x_0^z - (z_0 + \cdots + z_{k-1})) \right]$$

which is independent of the y_i.

Let us then, for $N \geq d$, introduce the sequence of functions defined by

$$\begin{aligned} f_N(x) = \operatorname*{Min}_{P} \operatorname*{Exp}_{z_i} \Bigg[\sum_{k=0}^{N-d} g(y_i) + \sum_{k=d-1}^{N-1} \phi(x_0 - (z_0 + z_1 + \cdots + z_k) \\ + y_0 + y_1 + \cdots + y_{k-d+1}) \Bigg]. \end{aligned} \tag{1}$$

The notation Min_p indicates that the minimization is over all policy functions.
We have

$$f_d(x) = \operatorname*{Min}_{y_0} \operatorname*{Exp}_{z_0, z_1, \cdots, z_{d-1}} [g(y_0) + \phi(x_0 - (z_0 + z_1 + \cdots + z_{d-1}) + y_0)]. \tag{2}$$

To obtain a recurrence relation connecting $f_N(x)$ and $f_{N-1}(x)$, observe that the effect of an initial choice of y_0 and an initial demand z_0 is to produce a similar sum with x_0 replaced by $x_0 + y_0 - z_0$. Taking account of the initial terms $g(y_0) + \operatorname{Exp}_{z_0} \phi(x_0 + y_0 - z_0)$, we expect to obtain the following recurrence relation upon using the principle of optimality, [2].

$$f_N(x) = \operatorname*{Min}_{y_0} \Bigg[g(y_0) + \operatorname{Exp} \phi(x_0 - (z_0 + \cdots + z_{d-1}) + y_0) + \int_0^\infty f_{N-1}(x_0 - z_0 + y)\, dH(z_0) \Bigg], \tag{3}$$

$N = d + 1, \cdots$.

5. Rigorous derivation. In order to derive this recurrence relation in a rigorous fashion, we must make precise what is meant by a sequence of ordering functions and what is meant by minimization over these functions.

The quantity y_i is a stochastic variable, dependent upon $x_0, z_0, z_1, \cdots, z_{i-1}$ and $y_0, y_1, \cdots, y_{i-1}$, or $y_i = y_r(z_0, z_1, \cdots, z_{i-1}, y_0, y_1, \cdots, y_{i-1}; x)$. The multistage nature of the process shows that $f_N(x)$ may be written

$$f_N(x) = \operatorname*{Min}_{y_0(x_0)} [\operatorname*{Exp}_{z_0} [\operatorname*{Min}_{y_1(z_0, y_0; x_0)} [\operatorname*{Exp}_{z_1} [\cdots]]]]. \tag{1}$$

It follows that we can write

$$f_N(x) = \operatorname*{Min}_{y_0(x_0)} [g(y_0) + \operatorname*{Exp}_{z} \phi(x_0 - (z_0 + z_1 + \cdots + z_{d-1}) + y_0) + \operatorname*{Exp}_{z_0} \operatorname*{Min}_{y_1(z_0, y_0; x_0)} [\operatorname*{Exp}_{z_1} [\cdots]]]. \tag{2}$$

The expression

$$g_{N-1} = \operatorname*{Exp}_{z_0} \operatorname*{Min}_{y_1(z_0, y_0; x_0)} [\operatorname*{Exp}_{z_1} [\cdots]] \tag{3}$$

looks like $\operatorname{Exp}_{z_0} f_{N-1}(x_0 + y_0 - z_0)$, with the difference that the minimization in (1) is over the functions $y_0(x_0)$, $y_1(z_0, y_0; x_0)$, and so on, while the minimization in (3) is over the functions $y_1(z_0, y_0; x_0)$, $y_2(z_0, z_1, y_0, y_1; x_0)$, and so on. Observe, however, in what fashion the variables z_0, y_0 and x_0 enter in (3). They occur only in the combination $x_0 + y_0 - z_0$. Consequently, minimization over $y_1(z_0, y_0; x_0)$ is equivalent to minimization over $y_1(x_0 + y_0 - z_0)$; similarly, minimization over $y_2(z_0, z_1, y_0, y_1; x_0)$ is equivalent to minimization over $y_2(z_1, y_1; x_0 + y_0 - z_0)$, and so on.

It follows that

$$g_{N-1} = \operatorname*{Exp}_{z_0} f_{N-1}(x_0 + y_0 - z_0). \tag{4}$$

This establishes the formula of (4.3).

6. Discussion. We see then that the solution of the original inventory problem can be treated in terms of a sequence of functions of one variable, regardless of the size of d. Under various simplifying assumptions concerning purchasing cost and outage cost, we can determine the structure of the optimal policy; see, for example, the discussion in [2], [9].

If, however, we take into account fixed costs which occur in realistic processes, formidable analytic difficulties arise. Nonetheless, the computational solution can readily be obtained using (4.3).

References

1. K. D. Arrow, T. E. Harris, and J. Marschak, *Optimal inventory policy*, *Econometrica* 19, 250–272 (1951)
2. R. Bellman, *Dynamic programming*, Princeton University Press, Princeton, New Jersey, 1957
3. R. Bellman, *Dynamic programming and the computational solution of feedback design control processes*, Conference on Computers in Control, AIEE, October 16–18, 1957, Atlantic City, New Jersey
4. R. Bellman, *Some new techniques in the dynamic programming solution of variational problems*, Quart. Appl. Math., **16** 295–305 (1958)
5. R. Bellman, *On a dynamic programming approach to the caterer problem—I*, Management Science **3**, 270–278 (1957)
6. R. Bellman, I. Glicksberg and O. Gross, *On the optimal inventory equation*, Management Science **2**, 83–104 (1955)
7. A. Dvoretzky, J. Kiefer and J. Wolfowitz, *The inventory problem, I, II*, *Econometrica* **20**, 187–222 (1952)
8. T. M. Whitin, *The theory of inventory management*, Princeton University Press, Princeton, New Jersey, 1953
9. H. Scarf, S. Karlin and K. J. Arrow, *Studies in the mathematical theory of inventory and production*, Stanford Univ. Press, 1958

AN ELEMENTARY DISCUSSION OF DEFINITIONS OF STRESS RATE*

BY WILLIAM PRAGER (*Brown University*)

1. Introduction. The simplest constitutive equation considered in the theory of plasticity only involves the tensors of stress and rate of deformation and described rigid, perfectly plastic behavior. When elastic effects are to be included in the analysis, this equation is assumed to apply to the plastic part of the rate of deformation, to which an elastic part must be added before the total rate of deformation is obtained. In a similar manner, a simple constitutive equation for a viscoelastic material can be established by adding an elastic rate of deformation to the rate of deformation of a viscous fluid. In both cases the elastic rate of deformation is usually written as a function of an appropriately defined rate of stress.

This stress rate must obviously satisfy the following condition: if a stressed continuum performs a rigid body motion and the stress field is independent of time when referred to a coordinate system that participates in this motion, the stress rate vanishes identically. As is readily seen, this restriction is not severe enough to lead to a unique definition of stress rate. Indeed, from one definition that satisfies this condition another one may be obtained by adding terms that contain the rate of deformation. In a rigid body motion, these terms vanish, and the second definition reduces to the first. Since the condition imposed on the definition of stress rate only concerns rigid body motions, it will be satisfied by the second definition if it is satisfied by the first. As a consequence of this freedom of choice, many definitions of stress rate are found in the literature.

A similarly embarrassing choice offers itself when one attempts to define finite strain. Potentially, any tensor formed from the displacement gradients qualifies as strain tensor if it vanishes identically for all rigid body motions. Depending on the field of

*Received Feb. 6, 1960. This paper is based on a report distributed under Contract Nonr 562(10) between the Office of Naval Research and Brown University.